Goal: If we know that a power series converges to a specific function, we can manipulate the equation to determine the limits of new power series. This is a nifty and fast way to get lots of new power series representations of functions. Today we will manipulate power series in these ways:

- Substitute
- Multiply by x
- Differentiate
- Integrate

1. Write down a power series representation for the function $f(x) = \dfrac{1}{1-x}$ by using the fact that the geometric series $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. Write your answer in both expanded form and Σ-notation. On what interval does the series converge to the function?

   Solution: Letting $r = x$ and $a = 1$ in the geometric series formula, we get

   $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + ... = \sum_{n=0}^{\infty} x^n$$

   From the Geometric Series Test, the equality is true for $-1 < x < 1$.

2. Using your response for the last problem, substituting $-x$ in the place of x, find the power series representation for $f(x) = \dfrac{1}{1+x}$. Write your answer in both expanded form and Σ-notation. On what interval does the series converge to the function?

   Solution:

   $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + ... + (-1)^n x^n + ... = \sum_{n=0}^{\infty} (-1)^n x^n$$

   From the Geometric Series test, we require $|-x| < 1$, so $-1 < x < 1$.

3. Find the power series representation for $f(x) = \dfrac{1}{1+x^2}$. Write your answer in both expanded form and Σ-notation. On what interval does the series converge to the function?

   Solution:

   $$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + ... + (-1)^n x^{2n} + ... = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

   From the Geometric Series test, we require $|-x^2| < 1$, so $-1 < x < 1$.

4. Find the power series representation for $\frac{x}{1-x}$. (Hint: multiply answer to problem 1 by x.) On what interval does the series converge to the function?

Solution: We start with $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + ... = \sum_{n=0}^{\infty} x^n$, so $\frac{x}{1-x} = x + x^2 + x^3 + ... + x^{n+1} + ... = \sum_{n=0}^{\infty} x^{n+1}$. Multiplying by x does not change the interval on which the equality holds, so the interval is $-1 < x < 1$.

5. Find the power series representation for $\frac{1}{(1-x)^2}$. On what interval does the series converge to the function? Hint: Take the derivative of both sides of this equation:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + ... = \sum_{n=0}^{\infty} x^n$$

Solution: $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$. Now we'll take the derivative of $1 + x + x^2 + x^3 + ... + x^n + ...$ term-by-term. We get

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + ... + nx^{n-1} + ... = \sum_{n=1}^{\infty} nx^{n-1}$$

Notice that the index on the summation starts at $n = 1$ now (the first term is gone, since it was the derivative of a constant). Differentiating does not change the radius of the interval, so we still have $-1 < x < 1$.

6. Find the power series representation of $\arctan x$. (Hint: start with the power series for $\frac{1}{1+x^2}$ and antidifferentiate. Solve for the constant of integration by substituting $x = 0$.) On what interval does the series converge to the function?

Solution: From problem 3, $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + ... + (-1)^n x^{2n} + ... = \sum_{n=0}^{\infty}(-1)^n x^{2n}$.
Antidifferentiating both sides gives:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... + \frac{(-1)^n x^{2n+1}}{2n+1} + ... + C = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C$$

Now substitute $x = 0$ into both sides, recalling that $\arctan 0 = 0$:

$$0 = 0 - 0 + 0 ... + C$$

So $C = 0$. We have:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... + \frac{(-1)^n x^{2n+1}}{2n+1} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Anti-differentiating does not change the radius of the interval, so we still have $-1 < x < 1$.