FINAL EXAM
CALCULUS 2

MATH 2300
FALL 2018

Name

PRACTICE EXAM

SOLUTIONS

Please answer all of the questions, and show your work.
You must explain your answers to get credit.
You will be graded on the clarity of your exposition!

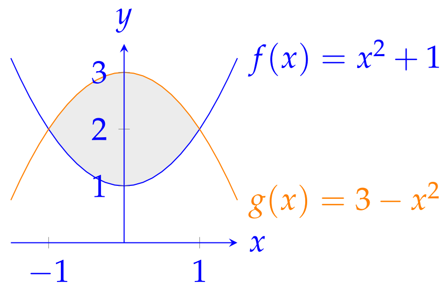
Date: December 12, 2018.

1
10 points

1. Consider the region bounded by the graphs of $f(x) = x^2 + 1$ and $g(x) = 3 - x^2$.

1.(a). *(5 points)* Write the integral for the volume of the solid of revolution obtained by rotating this region about the x-axis. Do not evaluate the integral.

SOLUTION: We can see the region in question below.



Using the washer method, the volume integral is

$$\pi \int_{-1}^{1} g(x)^2 - f(x)^2 \, dx = \pi \int_{-1}^{1} (3 - x^2)^2 - (x^2 + 1)^2 \, dx.$$

1.(b). *(5 points)* Write the integral for the volume of the solid of revolution obtained by rotating this region about the line $x = 3$. Do not evaluate the integral.

SOLUTION: Now using the shell method, the integral is equal to

$$\int_{-1}^{1} 2\pi(3 - x)(g(x) - f(x)) \, dx = 2\pi \int_{-1}^{1} (3 - x)((3 - x^2) - (x^2 + 1)) \, dx$$

$$= 2\pi \int_{-1}^{1} (3 - x)(2 - 2x^2) \, dx$$

2
6 points

2. MULTIPLE CHOICE: Circle the best answer.

2.(a). *(1 point)* Is the integral $\int_{-1}^{1} \frac{1}{x^2}\, dx$ an improper integral?

Yes No

2.(b). *(5 points)* Evaluate the integral: $\int_{-1}^{1} \frac{1}{x^2}\, dx =$

SOLUTION: The function $1/x^2$ is undefined at $x = 0$, so we we must evaluate the improper integral as a limit.

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{x^2}\, dx &= \lim_{c\to 0^-} \int_{-1}^{c} \frac{1}{x^2}\, dx + \lim_{c\to 0^+} \int_{c}^{1} -\frac{1}{x^2}\, dx \\
&= \lim_{c\to 0^-} -\frac{1}{x}\Big|_{-1}^{c} + \lim_{c\to 0^+} -\frac{1}{x}\Big|_{c}^{1} \\
&= \lim_{c\to 0^-} -\left(\frac{1}{c} - \frac{1}{-1}\right) + \lim_{c\to 0^+} -\left(\frac{1}{1} - \frac{1}{c}\right) \\
&= \lim_{c\to 0^-} -\left(\frac{1}{c} + 1\right) + \lim_{c\to 0^+} -\left(1 - \frac{1}{c}\right).
\end{aligned}$$

Now, since

$$\lim_{c\to 0^-} -\left(\frac{1}{c} + 1\right) = \lim_{c\to 0^-} \frac{-1}{c} - 1$$

and

$$\lim_{c\to 0^+} -\left(1 - \frac{1}{c}\right) = \lim_{c\to 0^+} \frac{1}{c} - 1$$

both diverge to ∞, and so the integral does not converge. Thus, the integral diverges.

3
14 points

3. Consider the curve parameterized by $\begin{cases} x = \frac{1}{3}t^3 + 3t^2 + \frac{2}{3} \\ y = t^3 - t^2 \end{cases}$ for $0 \le t \le \sqrt{5}$.

3.(a). *(6 points)* Find an equation for the line tangent to the curve when $t = 1$.

SOLUTION: We first find a general formula for the slope using the chain rule, and then evaluate at $t = 1$, giving

$$\left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right|_{t=1} = \left.\frac{3t^2 - 2t}{t^2 + 6t}\right|_{t=1} = \frac{1}{7}.$$

Since $x(1) = 4$ and $y(1) = 0$, we need the formula for a line with slope $1/7$ that passes through $(4, 0)$. This equation is

$$y = \frac{1}{7}x - \frac{4}{7}$$

3.(b). *(3 points)* Compute $\dfrac{d^2y}{dx^2}$ at $t = 1$.

SOLUTION: Again employing the chain rule,

$$\left.\frac{d^2y}{dx^2}\right|_{t=1} = \left.\frac{d}{dx}\frac{dy}{dx}\right|_{t=1} = \left.\frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}}\right|_{t=1} = \left.\frac{\frac{(6t-2)(t^2+6t)-(2t+6)(3t^2-2t)}{(t^2+6t)^2}}{t^2+6t}\right|_{t=1} = \frac{20}{7^3}.$$

3.(c). *(5 points)* Write an integral to compute the total arc length of the curve. Do not evaluate the integral.

SOLUTION: Arc length is given by

$$\int_0^{\sqrt{5}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\sqrt{5}} \sqrt{(t^2+6t)^2 + (3t^2-2t)^2}\, dt.$$

4
8 points

4. Consider the function $f(x) = x^2 \arctan(x)$.

4.(a). *(5 points)* Find a power series representation for $f(x)$.

SOLUTION: The power series of $\arctan(x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, with interval of convergence $x \in [-1, 1]$. Thus,

$$f(x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{2n+1}$$

for $x \in [-1, 1]$.

4.(b). *(3 points)* What is $f^{(83)}(0)$, the 83rd derivative of $f(x)$ at $x = 0$?

SOLUTION: For a power series $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ with positive radius of convergence, we have $f^{(n)}(a) = n! c_n$. In our power series representation $f(x) = x^2 \arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+3}$, which has radius of convergence 1, the coefficient of $x^{83} = x^{2 \cdot 40+3}$ is $\frac{(-1)^{40}}{2 \cdot 40+1} = \frac{1}{81}$, so that $f^{(83)}(x) = \frac{83!}{81}$.

Alternatively, using the above power series representation, and formally differentiating, we have

$$f^{(83)}(x) = \sum_{n=40}^{\infty} \frac{(2n+3)!}{(2n+3-83)!} \frac{(-1)^n x^{2n+3-83}}{2n+1} = \sum_{n=40}^{\infty} \frac{(2n+3)!}{(2(n-40))!} \frac{(-1)^n x^{2(n-40)}}{2n+1}.$$

Thus,

$$f^{(83)}(0) = (83)! \frac{(-1)^{40}}{2 * 40 + 1} = 83 * 82 * (80!).$$

5. A tank contains 200 L of salt water with a concentration of 4 g/L. Salt water with a concentration of 3 g/L is being pumped into the tank at the rate of 8 L/min, and the tank is being emptied at the rate of 8 L/min. Assume the contents of the tank are being mixed thoroughly and continuously. Let $S(t)$ be the amount of salt (measured in grams) in the tank at time t (measured in minutes).

5
10 points

5.(a). *(1 points)* What is the amount of salt in the tank at time $t = 0$?

SOLUTION: $S(0)\mathrm{g} = 200\mathrm{L} \cdot 4\mathrm{g/L} = 800$ g.

5.(b). *(2 points)* What is the rate at which salt enters the tank?

SOLUTION: $8\mathrm{L/min} \cdot 3\mathrm{g/L} = 24\ \mathrm{g/min}$

5.(c). *(2 points)* What is the rate at which salt leaves the tank at time t?

SOLUTION: As the volume of water is a constant 200 L, this is $\frac{S(t)\mathrm{g}}{200\mathrm{L}} \frac{8\mathrm{L}}{\mathrm{min}} = \frac{S(t)}{25} \frac{\mathrm{g}}{\mathrm{min}}$.

5.(d). *(1 points)* What is $\frac{dS}{dt}$, the net rate of change of salt in the tank at time t?

SOLUTION: Net change is given by gain minus loss, so using parts (b) and (c),

$$\frac{dS}{dt} \frac{\mathrm{g}}{\mathrm{min}} = 24 - \frac{S(t)}{25} \frac{\mathrm{g}}{\mathrm{min}}$$

5.(e). *(4 points)* Write an initial value problem relating $S(t)$ and $\frac{dS}{dt}$. Solve the initial value problem.

SOLUTION: The initial value problem is $\frac{dS}{dt} = 24 - \frac{S(t)}{25}$, with $S(0) = 800$. Since this differential equation is separable, we can solve by separating and then integrating:

$$\int \frac{1}{24 - \frac{1}{25}S}\, dS = \int dt$$

$$-25 \ln \left| 24 - \frac{1}{25}S \right| = t + C,$$

Note that $24 - \frac{1}{25}S \leq 0$, so we can write this as $-25 \ln\left(\frac{1}{25}S - 24\right) = t + C$, so that $\frac{1}{25}S - 24 = Ae^{-\frac{1}{25}t}$. From this we get $S = Ae^{-\frac{1}{25}t} + 600$. Setting $t = 0$, and using (a), we find the answer is

$$S = 200e^{-\frac{1}{25}t} + 600$$

6
8 points

6. Compute the following integrals.

6.(a). *(4 points)* $\int \sin^3(x)\cos^2(x)\,dx$

SOLUTION: First, using the pythagorean identity,

$$\begin{aligned}\int \sin^3(x)\cos^2(x)\,dx &= \int \sin(x)(1-\cos^2(x))\cos^2(x)\,dx \\ &= \int \sin(x)\cos^2(x)\,dx - \int \sin(x)\cos^4(x)\,dx.\end{aligned}$$

Now, let $u = \cos(x)$, so that $du = -\sin(x)\,dx$. Then the above equation is equal to

$$\int -u^2\,du + \int u^4\,du = -\frac{u^3}{3} + \frac{u^5}{5} + C.$$

Finally, reversing our substitution, we find that

$$\int \sin^3(x)\cos^2(x)\,dx = -\frac{\cos^3(x)}{3} + \frac{\cos^5(x)}{5} + C.$$

6.(b). *(4 points)* $\int \frac{x+1}{x^2(x-1)}\,dx$

SOLUTION: We start by using partial fractions:

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1},$$

which gives

$$x+1 = Ax(x-1) + B(x-1) + Cx^2 = (A+C)x^2 + (B-A)x - B,$$

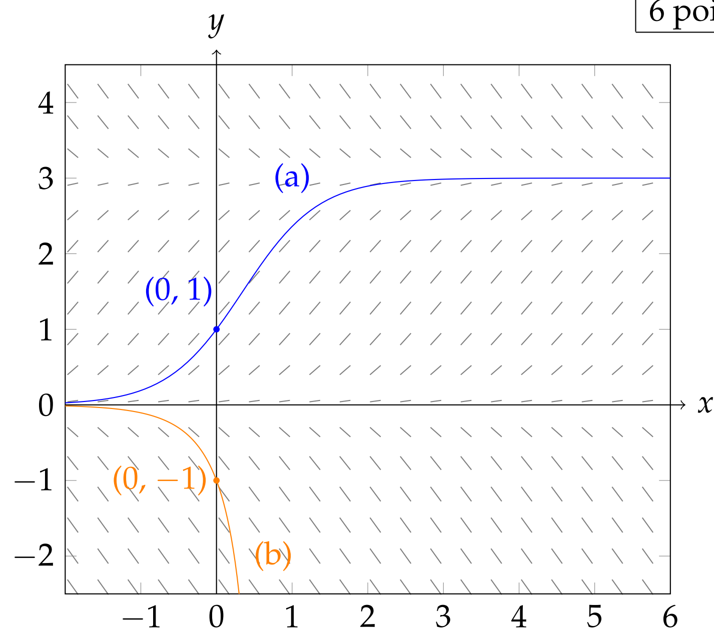
from which we deduce $A + C = 0$, $B - A = 1$, and $-B = 1$. Therefore, $B = -1$, $A = -2$, and $C = 2$. Thus,

$$\begin{aligned}\int \frac{x+1}{x^2(x-1)}\,dx &= \int \frac{-2}{x} + \frac{-1}{x^2} + \frac{2}{x-1}\,dx \\ &= -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C.\end{aligned}$$

7. A slope field for the differential equation $y' = 2y\left(1 - \frac{y}{3}\right)$ is shown below.

7
6 points

$$y' = 2y\left(1 - \frac{y}{3}\right)$$



7.(a). *(2 points)* Sketch the graph of the solution that satisfies following initial condition. Label the solution as (a).

$$y(0) = 1$$

7.(b). *(2 points)* Sketch the graph of the solution that satisfies following initial condition. Label the solution as (b).

$$y(0) = -1$$

7.(c). *(2 points)* Show that for $y(0) = c \geq 0$, we have $\lim_{x\to\infty} y(x)$ is finite.

SOLUTION: That this should be true is evident from the picture above. To see that it is in fact true, we argue as follows. First, if $P_0 = 0$, then $P(t) = P_0$ for all t, and $\lim_{t\to\infty} P(t) = 0$. If $P_0 \neq 0$, consider the general solution to the logistics equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \qquad P(t) = \frac{M}{1 + (\frac{M}{P_0} - 1)e^{-kt}}$$

The function $P(t)$ is well-defined, so long as the denominator is non-zero. We focus here on the case $k, M > 0$. If $0 < P_0 \leq M$, so that $(\frac{M}{P_0} - 1) \geq 0$, then the denominator is clearly never zero, and we have $\lim_{t\to\infty} P(t) = M$. If $P_0 > M$, then it is also easy to see that the denominator is never zero for $t \geq 0$, and so again, one easily computes $\lim_{t\to\infty} P(t) = M$.

Note however, that if $P_0 < 0$, then we have $1 + \left(\frac{M}{P_0} - 1\right) e^{-kt} = 0 \iff t = \frac{1}{k}\log\left(1 - \frac{M}{P_0}\right)$ In fact, it is not hard to check that for $P_0 < 0$, we have $\lim_{t\to\frac{1}{k}\log\left(1-\frac{M}{P_0}\right)} P(t) = -\infty$

8
6 points

8. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

8.(a). *(3 points)* Use the Remainder Estimate for the Integral Test to find an upper bound for the error in using S_{10} (the 10th partial sum) to approximate the sum of this series.

SOLUTION: If R_{10} denotes the error described above, the Remainder Estimate for the Integral Test tells us that

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^4}\, dx = \lim_{c\to\infty} \frac{1}{-3} \frac{1}{x^3}\bigg|_{10}^{c} = \lim_{c\to\infty} \frac{1}{-3} \frac{1}{c^3} - \frac{1}{-3} \frac{1}{10^3} = \frac{1}{3000}.$$

8.(b). *(3 points)* How many terms suffice to ensure that the sum is accurate to within 10^{-6}?

SOLUTION: In order to ensure that the error in estimate is less than 10^{-6} with the Remainder Estimate for the Integral Test (REIT), we must solve

$$\int_{N}^{\infty} \frac{1}{x^4}\, dx \leq \frac{1}{10^6}.$$

Following the solution in (a),

$$\int_{N}^{\infty} \frac{1}{x^4}\, dx = \lim_{c\to\infty} \frac{1}{-3} \frac{1}{c^3} - \frac{1}{-3} \frac{1}{N^3} = \frac{1}{3N^3},$$

and so we must solve

$$\frac{1}{3N^3} \leq \frac{1}{10^6},$$

So clearly it suffice to take $N = 10^2$. In other words, if we use 100 terms, we are ensured by the REIT that the error is at most 10^{-6}.

9
12 points

9. Determine whether the series is convergent or divergent and circle the corresponding answer. Then write the test allows one to determine convergence or divergence

9.(a). *(3 points)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}}$

convergent divergent

Test: p-series test

9.(b). *(3 points)* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{n^2-3}$

convergent divergent

Test: alternating series test

9.(c). *(3 points)* $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{5}{n}\right)$

convergent divergent

Test: test for divergence

9.(d). *(3 points)* $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+5}{(n+2)!}$

convergent divergent

Test: ratio test

10
6 points

10. MULTIPLE CHOICE: Circle the best answer below.

10.(a). *(2 points)* The sequence $a_n = 1 - 0.2^n$

converges to 0. converges, but not to 0. diverges.

10.(b). *(2 points)* The sequence $a_n = \dfrac{3n - 4}{2n - 1}$

converges to 0. converges, but not to 0. diverges.

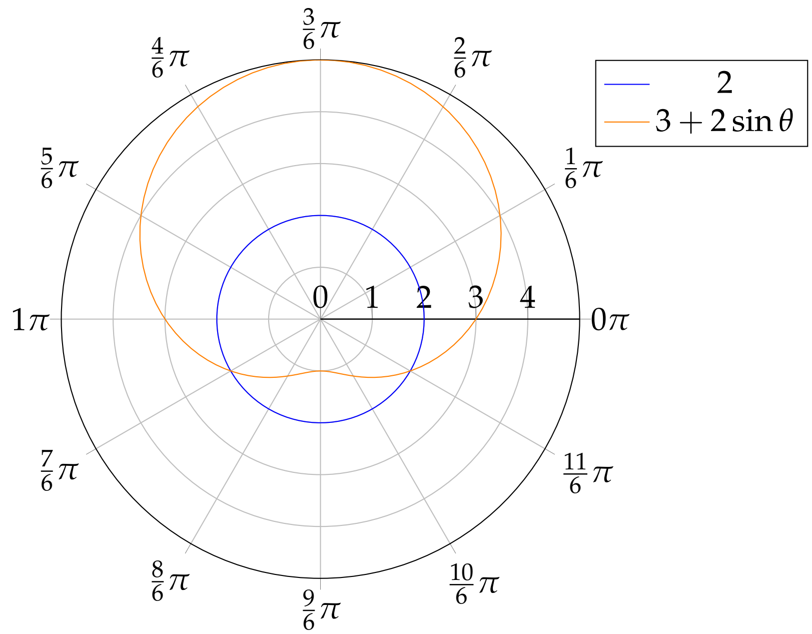
10.(c). *(2 points)* The sequence $a_n = n + \dfrac{1}{n}$

converges to 0. converges, but not to 0. diverges.

11
8 points

11.

11.(a). *(4 points)* Sketch the curves $r = 2$ and $r = 3 + 2\sin\theta$ on the axes below.



11.(b). *(4 points)* Write an integral that represents the area contained outside the first curve ($r = 2$) and inside the second curve ($r = 3 + 2\sin(\theta)$). Do not evaluate the integral.

SOLUTION: From the graph (or via algebraic solution), the bound of integration should be $-\pi/6$ to $7\pi/6$. Thus, the integral is

$$\frac{1}{2}\int_{-\pi/6}^{7\pi/6} (3 + 2\sin(\theta))^2 - 2^2 \, d\theta$$

12
8 points

12. MULTIPLE CHOICE: Circle the best answer below.

12.(a). *(2 points)* Is the following statement ALWAYS, SOMETIMES, or NEVER true?

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

ALWAYS SOMETIMES NEVER

12.(b). *(2 points)* Is the following statement ALWAYS, SOMETIMES, or NEVER true?

If $\sum a_n$ converges, then $\sum |a_n|$ converges.

ALWAYS SOMETIMES NEVER

12.(c). *(2 points)* The graph of $\begin{cases} x = t^2 - 3 \\ y = -t \end{cases}$ for $-\infty < t < \infty$ is a

line parabola circle ellipse

12.(d). *(2 points)* The graph of $\begin{cases} x = t^2 - 3 \\ y = -t^2 \end{cases}$ for $-\infty < t < \infty$ is a

line parabola circle ellipse