1. (6 points) Circle the correct answer.

(a) Determine the partial fraction decomposition of $\dfrac{x+3}{(x^2-1)(x+2)}$.

(i) $\dfrac{A}{x-1}+\dfrac{B}{x^2-1}+\dfrac{C}{x+2}$

(ii) $\dfrac{A}{x-1}+\dfrac{B}{x+2}$

(iii) $\dfrac{A}{x^2-1}+\dfrac{B}{x+2}$

(iv) $\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{C}{x+2}$

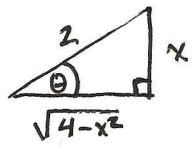
(b) Which trigonometric substitution is needed to integrate $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}dx$?

(i) $x = 2\tan(\theta)$

(ii) $x = 4\tan(\theta)$

(iii) $x = 2\sin(\theta)$

(iv) $x = 4\sin(\theta)$



$\sin\theta = \frac{x}{2}$

2. (8 points) Evaluate the following indefinite integral. Show all work.

$$\int \arctan(x)\,dx$$

$u = \tan^{-1}x \qquad v = x$

$du = \frac{1}{1+x^2}dx \qquad dv = dx$

$$\int u\,dv = uv - \int v\,du$$

$$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int \frac{x}{1+x^2}dx$$

$$= x\tan^{-1}x - \tfrac{1}{2}\ln(1+x^2) + C$$

Note Including absolute value for the logarithm is not necessary here as $x^2+1 \geq 0$ for all real numbers x.

3. (8 points) Compute the sum of

$$\sum_{n=1}^{\infty} \frac{5 \cdot 2^{n+1}}{3^{2n-1}}.$$

You do not need to simplify.

$$\frac{5 \cdot 2^{n+1}}{3^{2n-1}} = 5 \cdot 2 \left(\frac{2^n}{3^{2n-1}} \right)$$

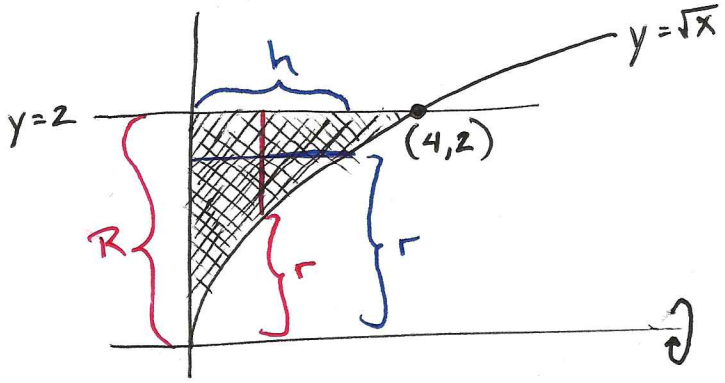
$$= 30 \left(\frac{2^n}{3^{2n}} \right) = 30 \left(\frac{2}{9} \right)^n$$

So this series is geometric w/ $a = 30 \cdot \left(\frac{2}{9}\right) = \frac{20}{3}$ (first term)

$$r = \frac{2}{9}$$

$$\Rightarrow \text{Sum is } \frac{a}{1-r} = \frac{20/3}{1-2/9} = \frac{9}{7} \cdot \frac{20}{3} = \frac{180}{21}$$

4. (8 points) Setup **but do not compute** an integral to find the volume of the solid of revolution determined by rotating the region bounded by $x = 0$, $y = 2$ and $y = \sqrt{x}$ about the x-axis.



Disk method (in red)

$R = 2$, $r = \{y\text{-coord of } y = \sqrt{x}\} = \sqrt{x}$

$$\Rightarrow V = \int_0^4 \pi (R^2 - r^2)\, dx$$

$$= \int_0^4 \pi (4 - x)\, dx$$

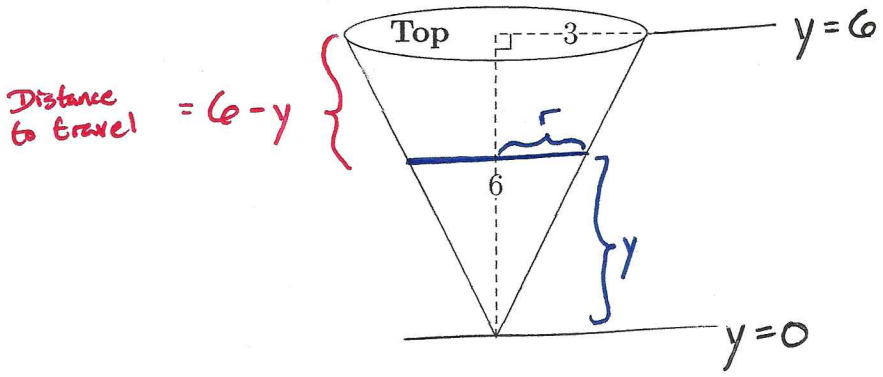
Shell Method (in blue)

$r = y$, $h = \{x\text{-coord of } y = \sqrt{x}\} = y^2$

$$V = \int_0^2 2\pi r h\, dy = \int_0^2 2\pi y^3\, dy$$

(Incidently, both integrals $= 8\pi$)

5. (8 points) Setup **but do not compute** an integral to find the work done pumping the water out of the top of a full tank with the shape of a right circular cone with height 6 meters, radius 3 meters. Express the acceleration due to gravity as g and the density of water as ρ.



$$W = \int \text{Force} * \text{Distance}$$

$$= \int (\text{mass} * g)(6-y)$$

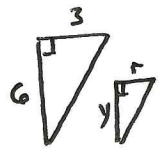
$$= \int (\rho * \text{volume})(g)(6-y)$$

$$= \int \rho g (\pi r^2 dy)(6-y)$$

as each blue slab of water is a disk w/ {area $= \pi r^2$, thickness $= dy$}

- Since the tank is full of water, the bounds of integration are from 0 to 6

$$= \int_0^6 \rho g \pi (r^2)(6-y)\,dy$$

To find how r & y relate, we use similar triangles 

$\Rightarrow r = \frac{1}{2}y$

$$\int_0^6 \rho g \pi \cdot \frac{1}{4} y^2 (6-y)\,dy$$

6. (8 points) Determine whether the series converges absolutely, converges conditionally, or diverges. Rigorously justify your answer.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

- To show convergence, we apply the Alternating Series Test.
- For $f(n) = \frac{1}{\sqrt{n+1}}$, the function f clearly ① goes to 0 as $n \to \infty$
  ② $f(n) \geq f(n+1)$ i.e. is decreasing
- So by the AST, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges.
- The series in absolute value $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ will be shown to diverge by the Limit Comparison Test w/ $b_n = \frac{1}{\sqrt{n}}$.
- Now both $\frac{1}{\sqrt{n+1}}, \frac{1}{\sqrt{n}} \geq 0$, and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, as seen with the p-test ($p = \frac{1}{2} \leq 1$).
- $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}$$
  $$= \sqrt{\lim_{n\to\infty} \frac{n}{n+1}} = \sqrt{1} = 1.$$
- As this limit is non-zero and $\sum_{n=1}^{\infty} b_n$ diverges, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges.
- Thus, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges conditionally

7. (8 points) Consider the function $f(x) = e^{3x^2}$.

(a) Write the Maclaurin series for $f(x)$.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\Rightarrow e^{3x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (3x^2)^n$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!} x^{2n}$$

(b) Use the Maclaurin series for $f(x)$ to express $\int_0^1 f(x)\,dx$ as a series.

$$\int_0^1 f(x)\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{3^n}{n!} x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \int_0^1 \frac{3^n}{n!} x^{2n}\,dx$$

(as the interval of conv. of $f(x)$'s Maclaurin series is all reals)

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!} \int_0^1 x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!} \frac{1}{2n+1} x^{2n+1} \Big|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{(2n+1)n!}$$

8. (6 points) Circle the correct answer.

(a) Find the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}} x^n$.

(i) $[-\frac{1}{3}, \frac{1}{3})$

(ii) $(-\frac{1}{3}, \frac{1}{3}]$

(iii) $(-\frac{1}{3}, \frac{1}{3})$

(iv) $[-\frac{1}{3}, \frac{1}{3}]$

(v) $(-\infty, \infty)$

~~The interval~~

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{3^n} = \lim_{n\to\infty} 3 \cdot \frac{\sqrt{n}}{\sqrt{n+1}} = 3$$

$\Rightarrow R = 1/3$

- If $x = 1/3$, the series is $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (diverges)
- If $x = -1/3$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ (converges)

(b) Using the power series

$$\ln(x+1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

and the Estimation Theorem for the Alternating Series, we conclude that the least number of terms in the series needed to approximate $\ln 2$ with error $< 3/1000$ is:

(i) 333

(ii) 534

(iii) 100

(iv) 9

(v) 201

If $x = 1$, we get $\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

The Alternating Series Estimation Thm states

$$\left| \text{Error in the partial sum } \sum_{n=0}^{N} a_n \text{ to approx the series} \right| \leq |a_{n+1}|$$

So we want $\frac{1}{n+1} < \frac{3}{1000} \Rightarrow n+1 > \frac{1000}{3} = 333.\overline{3}$

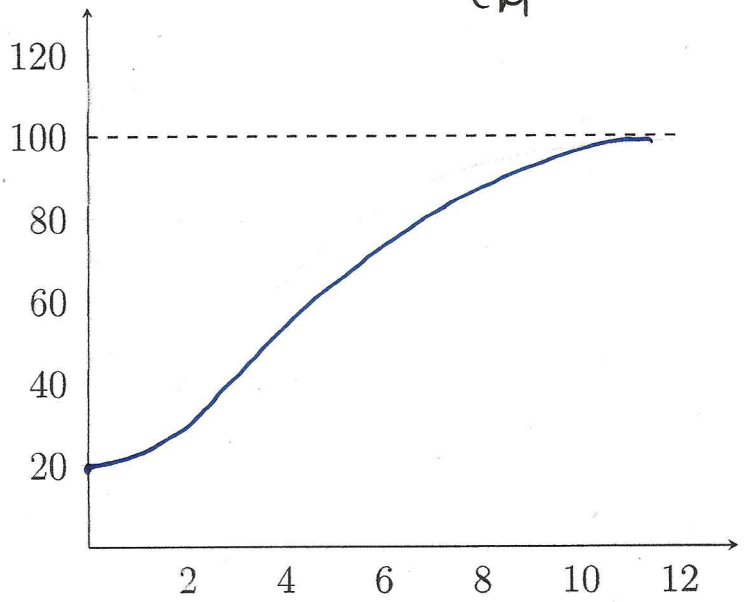
$\Rightarrow n+1 \geq 334$

$\Rightarrow n \geq 333$

9. (8 points) The following graph is a particular solution to the logistic differential equation

$$\frac{dP}{dt} = .5P\left(1 - \frac{P}{100}\right).$$

$100 \leftarrow M$



(a) Determine the carrying capacity, M, and the initial population, P_0.

$M = 100$ (also the asymptotic limit)

$P_0 = 20$ (y-intercept)

(b) Which function below represents the particular solution above?

(i) $P(t) = \dfrac{1}{1+4e^{-.5t}}$

(ii) $P(t) = \dfrac{100}{1+4e^{-.5t}}$

(iii) $P(t) = \dfrac{100}{1+20e^{-.5t}}$

(iv) $P(t) = \dfrac{1}{1+20e^{-.5t}}$

For $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$, the solution is $P = \frac{M}{1+ce^{-kt}}$

for some constant c.

Since $y(0)=20$, $20 = \frac{100}{1+c} \Rightarrow 1+c = \frac{100}{20} = 5 \Rightarrow c = 4$

10. (8 points) Consider the parametric curve

$$x(t) = t^2$$
$$y(t) = 2t + 2.$$

(a) Find the equation for the tangent line to the parametric curve at $t = 2$.

$x(2) = 4$

$y(2) = 6$

$$m = \left.\frac{dy/dt}{dx/dt}\right|_{t=2} = \left.\frac{2}{2t}\right|_{t=2} = \frac{1}{2}$$

$$y - 6 = \frac{1}{2}(x-4)$$

(b) Is the parametric curve concave up or down at $t = 2$? Justify your answer.

$$\frac{d^2y}{dx^2} = F'(t) \cdot \frac{1}{dx/dt} \quad \text{for } F(t) = \frac{dy}{dx}$$

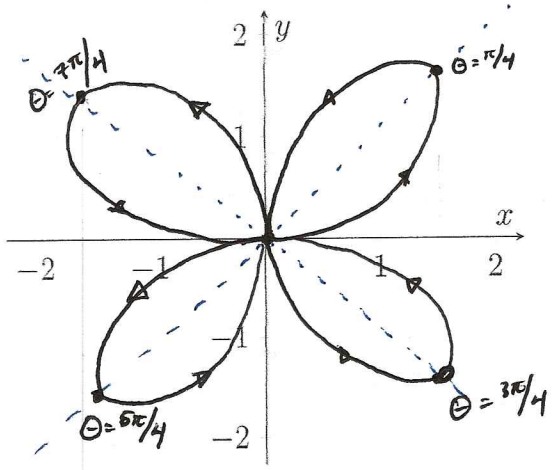
As above, $\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t} \Rightarrow F'(t) = -\frac{1}{t^2}$

$$\text{So } \frac{d^2y}{dx^2} = -\frac{1}{t^2} \cdot \frac{2t}{1} = -\frac{2}{t}$$

When $t = 2$, $\frac{d^2y}{dx^2} = -1 < 0$, so concave down.

11. (8 points) Follow the given steps to find the area of one petal of the rose curve given by $r = 2\sin(2\theta)$.

(a) Sketch the graph of $r = 2\sin(2\theta)$ on the interval $0 \leq \theta \leq 2\pi$.



θ	r
0	0
$\pi/4$	2
$\pi/2$	0
$3\pi/4$	−2
π	0
$5\pi/4$	2
$3\pi/2$	0
$7\pi/4$	−2
2π	0

(b) Fill in the boxes below to set up, **but not evaluate**, an integral which gives the area of one petal of the rose curve given by $r = 2\sin(2\theta)$.

$$\int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} \cdot \left(2\sin(2\theta)\right)^2 d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2\sin^2(2\theta)\, d\theta$$

$\int_{0}^{\pi/2}$ $2\sin^2(2\theta)\,d\theta$

12. (8 points) Determine whether the following improper integral converges. If the integral does converge, evaluate it.

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$$

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$$

$u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$

If $x = 1$, $u = 1$

$x = t$, $u = \sqrt{t}$

$$= \lim_{t\to\infty} \int_1^{\sqrt{t}} 2e^{-u}\,du$$

$$= \lim_{t\to\infty} -2e^{-u}\Big|_1^{\sqrt{t}}$$

$$= \lim_{t\to\infty} -2e^{-\sqrt{t}} + 2e$$

$$= 2e$$

The improper integral converges to $2e$.

13. (8 points) Find $\frac{dy}{dx}$ of the polar function $r = 2\cos(3\theta)$.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

for $y = r\sin\theta = 2\sin\theta\cos(3\theta)$

$x = r\cos\theta = 2\cos\theta\cos(3\theta)$

$$\frac{dy}{d\theta} = 2\cos\theta\cos(3\theta) - 6\sin\theta\sin(3\theta)$$

$$\frac{dx}{d\theta} = -2\sin\theta\cos(3\theta) - 6\cos\theta\cos(3\theta)$$

so $$\frac{dy}{dx} = \frac{2\cos\theta\cos(3\theta) - 6\sin\theta\sin(3\theta)}{-2\sin\theta\cos(3\theta) - 6\cos\theta\cos(3\theta)}$$