MATH 2300 – review problems for Exam 2

1. A metal plate of constant density ρ (in gm/cm^2) has a shape bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 1$.

   (a) Find the mass of the plate. Include units.

$$\text{Mass} = \text{Density} \times \text{Area} = \rho \times \int_0^1 \sqrt{x}\, dx = \frac{2}{3}\rho \text{ gm}$$

   (b) Find the center of mass of the plate. Include units.

$$\overline{x} = \frac{\rho \int_0^1 x\sqrt{x}\, dx}{\frac{2}{3}\rho} = \frac{3}{5} \text{ cm}, \qquad \overline{y} = \frac{\rho \int_0^1 \frac{1}{2}(\sqrt{x})^2\, dx}{\frac{2}{3}\rho} = \frac{3}{8} \text{ cm}$$

2. **(Exercise 7 from Section 6.6 in Stewart's Calculus Concepts and Contexts)** Suppose that 2J of work is needed to stretch a spring from its natural length of 30 cm to a length of 42 cm.

   (a) How much work is needed to stretch the spring from 35 cm to 40 cm?

   Force $= kx$ where k is the spring constant and x is the change from the natural length. Work $= \int_{x_0}^{x_1} kx dx$, thus for the spring in the problem

$$2 \text{ J} = \int_0^{.12} kx dx = .0072k \text{ m}^2 \Rightarrow k = \frac{2500}{9}\frac{\text{N}}{\text{m}^2}$$

.

$$W = \text{Work to stretch from 35 cm to 40cm} = \int_{.05}^{.1} kx dx = \frac{2500}{9}(.1^2 - .05^2) = \frac{25}{24} \text{ J.}$$

   (b) How far beyond its natural length will a force of 30 N keep the spring stretched?

   $F = kx$, 30 N $= \frac{2500}{9}$ N/m $x \Rightarrow x = 10.8$ cm.

3. **(Exercise 11 from Section 6.6 in Stewart's Calculus Concepts and Contexts)** Show how to approximate the required work by a Riemann sum. Then express the work as an integral and evaluate it: a heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft high.

   (a) How much work is done in pulling the rope to the top of the building?
   (b) How much work is done in pulling half the rope to the top of the building?

   Find the answers to exercises 6-8 (and all odd-numbered exercises in the textbook) in the back of the textbook.

4. **(Exercise 15 from Section 6.6 in Stewart's Calculus Concepts and Contexts)** Show how to approximate the required work by a Riemann sum. Then express the work as an integral and evaluate it: a leaky 10-kg bucket is lifted from the ground to a height of 12m at a constant speed with a rope that weighs 0.8 kg/m. Initially the bucket contains 36 kg of water, but the water leads at a constant rate and finishes draining just as the bucket reaches the 12-m level. How much work is done?

   Find the answers to exercises 6-8 (and all odd-numbered exercises in the textbook) in the back of the textbook.

5. Complete Exercise 19 from Section 6.6 on pg. 473 in Stewart's Calculus Concepts and Contexts

   Find the answers to exercises 6-8 (and all odd-numbered exercises in the textbook) in the back of the textbook.

6. Find the limit of all the sequences in the sequence activity:

   `http://math.colorado.edu/math2300/projects/SequencesPractice.pdf`

$$\lim_{n\to\infty} \frac{5}{2n+1} = 0$$
$$\lim_{n\to\infty} \frac{e^n}{n!} = 0$$
$$\lim_{n\to\infty} \sin(\frac{1}{n}) = \sin(0) = 0$$
$$\lim_{n\to\infty} \frac{(-1)^n n}{e^n} = 0$$
$$\lim_{n\to\infty} \frac{\sin(n^2)}{\sqrt{n}} = 0 \qquad \text{(Bound } \sin(n^2))$$
$$\lim_{n\to\infty} \frac{(-1)^n n^4}{n!} = 0 \qquad \text{Hint } (n! = n(n-1)(n-2)(n-3)(n-4)!)$$
$$\lim_{n\to\infty} \frac{n8^n}{3^{2n+1}} = \lim_{n\to\infty} \frac{n}{3}\left(\frac{8}{9}\right)^n = 0$$
$$\lim_{n\to\infty} \cos\left(\frac{n}{n^2}\right) = \cos(0) = 1$$
$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$
$$\lim_{n\to\infty} \frac{\ln n}{\ln(n^2)} = \lim_{n\to\infty} \frac{\ln n}{2\ln n} = \frac{1}{2}$$
$$\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{3n-1} = \frac{1}{3}$$
$$\lim_{n\to\infty} \frac{3n^3+n}{1-4n^2} = -\infty$$
$$\lim_{n\to\infty} \frac{(-1)^n 2^n}{n^4} \text{ does not converge.}$$
$$\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+1} \text{ does not converge.}$$
$$\lim_{n\to\infty} \frac{n}{\ln n} = \infty$$
$$\lim_{n\to\infty} \frac{10^n}{n5^n} = \infty$$
$$\lim_{n\to\infty} \frac{n^2}{n\ln n} = \infty$$
$$\lim_{n\to\infty} \frac{e^n}{\sqrt(n)} = \infty$$

7. Does $\{a_n\}$, where $a_n = \frac{1}{n}$, converge? If so, what does it converge to? I won't be fooled - a_n is a sequence, not a series. I am just being asked if the sequence has a limit. Yes, it converges to 0. $\lim_{n\to\infty} \frac{1}{n} = 0$. If I had been asked about the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n}$, then that answer would have been divergent, since it is the harmonic series.

8. Decide whether each of the following sequences converges. If a series converges, what does it converge to? If not, why not?

   (a) The sequence whose n-th term is $a_n = 1 - \frac{1}{n}$. Converges to 1. (The $\frac{1}{n}$ part goes to zero.)

(b) The sequence whose n-th term is $b_n = \sqrt{n+1} - \sqrt{n}$. Converges to 0:

$$\sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n}) \left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right) = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}},$$

and the denominator of this expression grows as n gets large, so it approaches zero.

(c) The sequence whose n-th term is $c_n = \cos(\pi n)$. Diverges. The sequence is $(-1, 1, -1, 1, \dots)$ which oscillates.

(d) The sequence $\{d_n\}$, where $d_1 = 2$ and

$$d_n = 2d_{n-1} \quad \text{for } n > 1.$$

Diverges. The sequence is $(2, 4, 8, 16, \dots)$, a geometric sequence with $r > 1$, so the terms go to infinity.

9. Find the sum of the series. For what values of the variable does the series converge to this sum?

(a) $1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \cdots$

(a) $\frac{2}{2-x}$, for $|x| < 2$

(b) $y - y^2 + y^3 - y^4 + \cdots$

(b) $\frac{y}{y+1}$, for $|y| < 1$

(c) $4 + z + \frac{z^2}{3} + \frac{z^3}{9} + \cdots$

(c) $\frac{z-12}{z-3}$, for $|z| < 3$

10. For each of the following series, determine whether or not they converge. If they converge, determine what they converge to.

(a) $\sum_{n=1}^{\infty} 5 \left(\frac{2}{3} \right)^{n-1}$ This is a geometric series with first term $a = 5$ and ratio $r = \frac{2}{3}$. $|r| < 1$, so by the geometric series test the series converges to $\frac{a}{1-r} = \frac{5}{1-\frac{2}{3}} = 15$.

(b) $\sum_{n=2}^{\infty} 3 \frac{4^{n+1}}{5^{n-4}}$ This is a geometric series. To get the first term I substitute $n = 2$ to get $a = 3 \cdot 4^3 / 5^{-2} = 3 \cdot 64 \cdot 25 = 4800$. The ratio is $r = \frac{4}{5}$. $|r| < 1$ so this series converges by the geometric series test to $\frac{a}{1-r} = \frac{4800}{1-\frac{4}{5}} = 24000$.

(c) $\sum_{n=3}^{\infty} \frac{7(-\pi)^{2n-1}}{e^{3n+1}} = \sum_{n=3}^{\infty} \frac{7(-\pi)^{2n}}{-\pi \cdot e \cdot e^{3n}} = \sum_{n=3}^{\infty} \frac{7((-\pi)^2)^n}{-\pi \cdot e \cdot (e^3)^n} \sum_{n=3}^{\infty} \frac{7(\pi^2)^n}{-\pi \cdot e \cdot (e^3)^n}$. Written in this form it is clear that this is a geometric series. Substitute $n = 3$ to find that the first term is $a = \frac{-7\pi^5}{e^7}$. $r = \frac{\pi^2}{e^3} < 1$, so by the Geometric Series Test, this series converges to $\frac{a}{1-r} = \frac{-7\pi^5}{e^7} \cdot \frac{1}{1-\frac{\pi^2}{e^3}}$

(d) $\sum_{n=2}^{8} 4(.07)^{n+1}$ This is a finite geometric series with the first term $a = 4(.07)^3$ and $r = .07$. The sum is $\frac{\text{first term}-\text{first unadded term}}{1-r} = \frac{4(.07)^3 - 4(.07)^{10}}{1-.07}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$. Using partial fractions decomposition, the series equals $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$.
Calculating partial sums for this telescoping series: $s_1 = \frac{1}{2}-\frac{1}{3}$, $s_2 = \frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4} = \frac{1}{2}-\frac{1}{4}$.
More generally $s_n = \frac{1}{2}-\frac{1}{n+2}$. Taking limits, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right) = \frac{1}{2}$

(See separate file for solutions to the next three problems)

11. For each of the following series, determine if it converges absolutely, converges conditionally, or diverges. Completely justify your answers, including all details.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n+5}$ $b_n = \frac{1}{n+5} > 0$,
$\frac{1}{n+1+5} < \frac{1}{n+5}$,
$\lim_{n\to\infty} b_n = 0$ so by the alternating series test $\sum_{n=2}^{\infty} \frac{(-1)^n}{n+5}$ converges. Let $a_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{n+5}{n} = 1$, therefore by the limit comparison test either both converge or both diverge. Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges $\sum_{n=2}^{\infty} \frac{1}{n+5}$ diverges. Therefore $\sum_{n=2}^{\infty} \frac{(-1)^n}{n+5}$ converges conditionally.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ Let $f(x) = \frac{1}{x(\ln x)^2}$ which is continuous and positive. f is decreasing.

$$\begin{aligned}
\int_2^{\infty} f(x)dx &= \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}dx \qquad\qquad u = \ln x, du = \frac{1}{x}dx \\
&= \lim_{b\to\infty} \int_{\ln 2}^b \frac{1}{u^2}du \\
&= \lim_{b\to\infty} \left.\frac{-1}{u}\right|_{\ln 2}^b = \frac{1}{\ln 2}.
\end{aligned}$$

Therefore by the integral test $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges absolutely.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(\ln n)^2}$, $b_n = \frac{n}{(\ln n)^2}$ is positive. $\ln n < n$, thus $\frac{1}{n} < \frac{n}{(\ln n)^2}$. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, and thus by the term size comparison test $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(\ln n)^2}$ diverges.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2(-3)^n}{n!}$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{2(n+1)^2(-3)^{n+1}}{(n+1)!}\frac{n!}{2n^2(-3)^n}\right| \\
&= \lim_{n\to\infty} \frac{2(n+1)^2}{n^2}\frac{3}{n+1} \\
&= 0 < 1
\end{aligned}$$

therefore by the ratio test $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2(-3)^n}{n!}$ converges absolutely.

(e) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4\cdot 2^n}{(-3)^{n+1}}+\frac{1}{2^n}\right)$

$$\sum_{n=1}^{\infty}\left(\frac{4\cdot 2^n}{(-3)^{n+1}}+\frac{1}{2^n}\right)=\sum_{n=1}^{\infty}\frac{4\cdot 2^n}{(-3)^{n+1}}+\sum_{n=1}^{\infty}\frac{1}{2^n}$$

both of which are geometric series, the first of which with $r=\frac{2}{-3}$, the second of which with $r=\frac{1}{2}$, both of which are less than 1 in absolute value. Therefore each converges absolutely, and the sum of two absolutely convergent series converges absolutely, thus $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4\cdot 2^n}{(-3)^{n+1}}+\frac{1}{2^n}\right)$ converges absolutely.

(f) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n}}{n^3+2n}$ Let $b_n=\left|\frac{(-1)^n\sqrt{n}}{n^3+2n}\right|$. $b_n<\frac{1}{n^{5/2}}$, $\sum_{n=1}\frac{1}{n^{5/2}}$ converges by the p test, and thus by the term size comparison test $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n}}{n^3+2n}$ converges absolutely.

(g) $\displaystyle\sum_{n=1}^{\infty}\frac{n+3n^5}{2n^7+3}$,$a_n=\frac{n+3n^5}{2n^7+3}>0$,$b_n=\frac{n^5}{n^7}=\frac{1}{n^2}>0$,

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{3}{2}.$$

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p test, and thus by the limit comparison test $\displaystyle\sum_{n=1}^{\infty}\frac{n+3n^5}{2n^7+3}$ absolutely.

(h) $\displaystyle\sum_{n=1}^{\infty}n^{\frac{1}{n}}$, let $L=\lim_{n\to\infty}n^{1/n}$.

$$\begin{aligned}\ln L &= \lim_{n\to\infty}\ln n^{1/n}\\ &= \lim_{n\to\infty}\frac{\ln n}{n}\\ &= \lim_{x\to\infty}\frac{\ln x}{x} && (\text{"}\frac{\infty}{\infty}\text{"})\\ &= \lim_{x\to\infty}\frac{1}{x} && \text{L'Hôspital's rule}\\ &= 0\end{aligned}$$

and thus $L=e^0=1$. Thus $\lim_{n\to\infty}n^{1/n}=1\neq 0$ and by the test for divergence $\displaystyle\sum_{n=1}^{\infty}n^{\frac{1}{n}}$ diverges.

(i) $\displaystyle\sum_{n=1}^{\infty}\arctan n$. $\lim_{n\to\infty}\arctan n=\frac{\pi}{2}\neq 0$ and thus by the test for divergence $\displaystyle\sum_{n=1}^{\infty}\arctan n$ diverges.

(j) $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$,$0\leq\left|\frac{\sin n}{n^2}\right|\leq\frac{1}{n^2}$. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p test, so $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ converges absolutely by the term size comparison test.

(k) $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^n$, let $L=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$

$$\begin{aligned}
\ln L &= \lim_{n\to\infty}\ln\left(\frac{n}{n+1}\right)^n\\
&= \lim_{n\to\infty} n\ln\left(1-\frac{1}{n+1}\right)\\
&= \lim_{n\to\infty}\ln\left(1-\frac{1}{n+1}\right)\Big/\frac{1}{n}\\
&= \lim_{x\to\infty}\ln\left(1-\frac{1}{x+1}\right)\Big/\frac{1}{x} && (\text{"}\frac{0}{0}\text{"})\\
&= \lim_{x\to\infty}\frac{-x^2}{(1-\frac{1}{x+1})(x+1)^2} && \text{L'Hôspital's rule}\\
&= -1
\end{aligned}$$

and thus $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=e^{-1}\neq 0$ and by the test for divergence $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^n$ diverges.

(l) $\sum_{n=1}^{\infty}\frac{n!}{n^n}$,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!}\right| &= \lim_{n\to\infty}(n+1)\frac{n^n}{(n+1)^{n+1}}\\
&= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\\
&= e^{-1} && (\text{by (k)})\\
&< 1
\end{aligned}$$

and thus $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges absolutely by the ratio test.

12. Consider the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$

(a) Confirm using the Alternate Series Test that the series converges.

$b_n=\frac{1}{n!}>0$, $\frac{1}{(n+1)!}<\frac{1}{n!}$, and $\lim_{n\to\infty}b_n=0$. Thus by the alternating series test $\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$ converges.

(b) How many terms must be added to estimate the sum to within .0001?

If $\sum_{n=1}^{\infty}(-1)^n b_n$ is a convergent alternating series then $|\sum_{n=1}^{\infty}(-1)^n b_n-\sum_{n=1}^{N}(-1)^n b_n|<b_{N+1}$.

$\frac{1}{8!}<.0001<\frac{1}{7!}$, and thus 7 terms are needed to estimate $\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$ to within .0001.

(c) Estimate the sum to within .0001.

13. How many terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ should be added to estimate the sum to within .01? No calculators.

$$\begin{aligned} b_{N+1} &< .01 \\ \frac{1}{\sqrt{N+1}} &< .01 \\ 100 &< \sqrt{N+1} \\ 9999 &< N \end{aligned}$$

thus you need 10000 terms to estimate within .01.

14. Check whether the following series converge or diverge. In each case, give the answer for convergence, and name the test you would use. If you use a comparison test, name the series $\sum b_n$ you would compare to.

(a) $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)}$ diverges – use integral or limit comparison test, comparing to $\frac{1}{n}$

(b) $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ converges – term-size comparison test with $\frac{1}{n^3}$

(c) $\sum_{n=1}^{\infty} \left(n + \frac{1}{n}\right)^n$ diverges – divergence test (nth term test) (that is, the n the term does not go to zero)

(d) $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{5n^2}$ diverges – limit comparison test or nth term test

(e) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ (hint: consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$) converges – limit comparison test

(f) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges – ratio test

(g) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n+3)!}$ diverges – ratio test

(h) $\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$ converges – simplify to $\frac{1}{(n+1)(n+2)}$ and then use term-size comparison test or limit comparison test.

(i) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges – ratio test

15. Consider the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$. Are the following statements true or false? Fully justify your answer.

(a) The series converges by limit comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n}$. False

(b) The series converges by the ratio test. False

(c) The series converges by the integral test. False

16. Consider the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$. Are the following statements true or false? Fully justify your answer.

(a) The series converges by limit comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n}$.False

(b) The series converges by the ratio test.False

(c) The series converges by the integral test. False

(d) The series converges by the alternating series test.True

(e) The series converges absolutely. False

17. Suppose the series $\sum a_n$ is absolutely convergent. Are the following true or false? Explain.

(a) $\sum a_n$ is convergent.true, because if a series converges absolutely, it must converge

(b) The sequence a_n is convergent. true, because $\sum a_n$ converges, so by the Divergence Test, $\lim_{n\to\infty} = 0$.

(c) $\sum (-1)^n a_n$ is convergent.True, in fact it converges absolutely.

(d) The sequence a_n converges to 1. False, the sequence converges to 0 by the Divergence Test.

(e) $\sum a_n$ is conditionally convergent. False

(f) $\sum \frac{a_n}{n}$ converges. True, this can be shown using the term-size comparison test.

18. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$$

You must justify your answer to receive credit. Yes, by the ratio test, because, if a_n denotes the nth term of this series, then the limit of $|a_{n+1}/a_n|$ as $n \to \infty$ is zero (which is less than 1).

19. A ball is dropped from a height of 10 feet and bounces. Assume that there is no air resistance. Each bounce is $\frac{3}{4}$ of the height of the bounce before.

(1) Find an expression for the height to which the ball rises after it hits the floor for the nth time.

$H(n) = 10(\frac{3}{4})^n$

(2) Find an expression for the total vertical distance the ball has traveled when it hits the floor for the nth time.

$D(n) = 10 + (2 \cdot 10 \cdot (3/4))\frac{1-(3/4)^{n-1}}{1-(3/4)}$

(3) Using without proof the fact that a ball dropped from a height of h feet reaches the ground in $\sqrt{h}/4$ seconds: Will the ball bounce forever? If not, how long it will take for the ball to come to rest?

The ball will not bounce forever. The total time it bounces is given by

$$(\sqrt{10}/4) + a/(1-r)$$

with $a = (1/2)\sqrt{10(3/4)}$ and $r = \sqrt{3/4}$.

Want more practice? Here's some more!

20. In theory, drugs that decay exponentially always leave a residue in the body. However in practice, once the drug has been in the body for 5 half-lives, it is regarded as being eliminated. If a patient takes a tablet of the same drug every 5 half-lives forever, what is the upper limit to the amount of drug that can be in the body?

    Let P_n represent the percentage of the drug in the body after the nth tablet. Then

$$P_1 = 1(100 \text{ percent})$$
$$P_2 = 1(.5 * .5 * .5 * .5 * .5) + 1 = 1.03125$$
$$P_3 = 1.03125(.03125) + 1 = 1.03223$$

    So,

$$P_n = \frac{1 - (.03125)^n}{1 - .03125}.$$

    As $n \to \infty$, $P_n \to 1.0323$, so this is the maximum amount of the drug in the body.

21. Let $\{f_n\}$ be the sequence defined recursively by $f_1 = 5$ and $f_n = f_{n-1} + 2n + 4$.

    (a) Check that the sequence g_n whose n-th term is $g_n = n^2 + 3n + 1$ satisfies this recurrence relation, and that $g_1 = 5$. (This tells us $g_n = f_n$ for all n.)
    We check:

$$g_{n-1} + 2n + 4 = ((n-1)^2 + 5(n-1) - 1) + 2n + 4 = n^2 + 5n - 1 = g_n$$

    and plainly $g_1 = 5$

    (b) Use the result of part (a) to find f_{20} quickly.

$$f_{20} = 20^2 + 5 \cdot 20 - 1 = 499$$

22. Find the values of a for which the series converges/diverges:

    (a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2a}\right)^n$

    (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a}\left(\frac{1}{2}\right)^n$

    (c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^a$

    (d) $\displaystyle\sum_{n=1}^{\infty} (\ln a)^n$

    (e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^a}$

(f) $\sum_{n=1}^{\infty} (1 + a^n)$

(g) $\sum_{n=1}^{\infty} (1 + a)^n$

(h) $\sum_{n=1}^{\infty} n^{\ln a}$

(i) $\sum_{n=1}^{\infty} a^{\ln n}$

(a) $|a| > 1/2$

(b) $a \neq 0$

(c) $a > 1$

(d) $e^{-1} < a < e$

(e) $a > 1$

(f) diverges for all a

(g) $-2 < a < 0$

(h) $0 < a < e^{-1}$

(i) $0 < a < e^{-1}$

23. Using the table below, estimate the length of the curve given by $y = f(x)$ from $(3, 4)$ to $(6, 0.7)$.

x	3	3.5	4	4.5	5	5.5	6
$f(x)$	4	3.6	2.4	-1	-0.5	0	0.7
$f'(x)$	-0.8	-2.4	-6.8	1	1	1.4	-0.4

The appropriate arc length formula is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

So, we can find a good estimate of the total length by using the table

$$\begin{aligned} L \approx &\sqrt{1 + (-0.8)^2}(.5) + \sqrt{1 + (-2.4)^2}(.5) + \sqrt{1 + (-6.8)^2}(.5) \\ &+ \sqrt{1 + (1)^2}(.5) + \sqrt{1 + (1)^2}(.5) + \sqrt{1 + (1.4)^2}(.5) \end{aligned}$$

24. Determine if these sequences converge absolutely, converge conditionally or diverge.

(a) $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges absolutely, take the absolute value and compare to $\frac{1}{n^2}$ using the term-size comparison test

(b) $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$ converges conditionally. Take the absolute value, then use limit comparison, comparing to $\frac{1}{n}$ to show it does not converge absolutely. Use alternating series test to show original series converges (and thus converges conditionally)

25. A steady wind blows a kite due east. The kite's height above ground from horizontal position $x = 0$ to $x = 80$ feet is given by

$$y = 150 - \frac{1}{40}(x-50)^2$$

Find the distance traveled by the kite. Just set up the integral - don't evaluate.

This distance can be calculated by finding the arc length of the above curve from $x = 0$ to $x = 80$. Using the arc length formula we find the distance traveled to be

$$L = \int_0^{80}\sqrt{1+(-\frac{1}{20}(x-50))^2}\,dx$$

For each of the following statements, determine if it is true Always, Sometimes or Never.

26. If a sequence a_n converges, then the sequence $(-1)^n a_n$ also converges. Sometimes. Try $a_n = 1$ and $a_n = 0$.

27. If a sequence $(-1)^n a_n$ converges to 0, then the sequence a_n also converges to 0. Always

28. The average value of a function is negative. Sometimes. Try $f(x) = 1$ on $[0,1]$, and $f(x) = -1$ on $[0,1]$.

29. The geometric series $\sum_{n=1}^{\infty} 5r^{n-1}$ converges to $\frac{5}{1-r}$ Sometimes - true if $|r| < 1$, false if $|r| \geq 1$.

30. The geometric series $\sum_{n=1}^{\infty}\frac{c}{5^{n-1}}$ converges to $\frac{5c}{4}$ Always, by the geometric series test.

31. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty}|a_n|$ converges. Sometimes. Try $a_n = (-1)^n/n^2$, and $a_n = (-1)^n/n$.

32. If a series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$ Always, by the divergence test.

33. If the series $\sum_{n=1}^{\infty}(-1)^n a_n$ is conditionally convergent, then the sequence a_n converges. Always, The series converges and so by the divergence test, $\{(-1)^n a_n\}$ must converge to 0, and so $\{a_n\}$ also converges to 0.

34. If the series $\sum_{n=1}^{\infty}(-1)^n a_n$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is convergent. Sometimes. Try $\sum(-1)^n/n$ and $\sum(-1)^n/n^2$.

35. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then $\sum_{n=1}^{\infty} a_n$ converges conditionally. Sometimes. Try $a_n = (-1)^n/n$ (cond. conv.) and $a_n = (-1)^n/n^2$ (abs. conv.) and $a_n = (-1)^n \cdot n$ (div.)

36. If the series $\sum_{n=1}^{\infty} |a_n|$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ diverges. Sometimes. Try $\sum(-1)^n/n$ and $\sum 1/n$