# MIDTERM 1
# CALCULUS 2

MATH 2300
FALL 2018

Monday, September 24, 2018
5:15 PM to 6:45 PM

Name ______________________________

**PRACTICE EXAM**

**SOLUTIONS**

Please answer all of the questions, and show your work.
You must explain your answers to get credit.
**You will be graded on the clarity of your exposition!**

*Date*: September 25, 2018.

**1.** Evaluate the following integrals:

| 1 |
|---|
| 80 points |

**1.(a).** $\int x \ln x dx$ $\boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$

**1.(b).** $\int x^3 e^{x^2} dx$ [Hint: Make a substitution first.] $\boxed{\frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C}$

**1.(c).** $\int_0^1 \frac{1}{(x^2+1)^{3/2}} dx$ $\boxed{\frac{\sqrt{2}}{2}}$

**1.(d).** $\int \cos(\ln x) dx$ [Hint: Use integration by parts.] $\boxed{\frac{1}{2}x\cos(\ln x) + \frac{1}{2}x \sin(\ln x) + C}$

**1.(e).** $\int 2\sin 2x \cdot (\sin x)^2 dx$ $\boxed{(\sin x)^4 + C}$

**1.(f).** $\int \frac{5x-1}{(x-1)(x-2)}$ $\boxed{-4\ln|x-1| + 9\ln|x-2| + C}$

**1.(g).** $\int \sqrt{25-x^2} dx$ $\boxed{\frac{25}{2}(\arcsin\frac{x}{5}) + \frac{1}{2}x\sqrt{25-x^2}) + C}$

**1.(h).** $\int \frac{2x^2+x-1}{(x-3)(x^2+1)}$ $\boxed{2\ln|x-3| + \arctan x + C}$

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SOLUTION

Here are more details on the solutions:

**1.(a).** $\int x \ln x dx$. We use Integration by Parts: ($\int u dv = uv - \int v du$). Let $u = \ln x$, $dv = x dx$, so that $du = \frac{1}{x} dx$ and $v = \frac{1}{2}x^2$. Then

$$\int x \ln x dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int \frac{x^2}{x} dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x dx = \boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$$

**1.(b).** $\int x^3 e^{x^2} dx$ [Hint: Make a substitution first.] We write $\int x \cdot x^2 e^{x^2} dx$. Now make a $u$-substitution I will use the letter $y$ instead of $u$ to avoid confusion later on: Let $y = x^2$ so that $dy = 2xdx$ (and $dx = \frac{dy}{2x}$). Substituting into the above integral, we have $\int x \cdot ye^y \frac{dy}{2x} = \int \frac{y}{2} e^y dy$ (Now use integration by parts) Let $u = y$, $dv = e^y dy$, so that $du = dy$ and $v = e^y$. Now we have

$$\int \frac{y}{2} e^y dy = \frac{1}{2} \int ye^y dy = \frac{1}{2} ye^y - \frac{1}{2} \int e^y dy = \frac{1}{2} ye^y - \frac{1}{2} e^y + C = \boxed{\frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C}$$

**1.(c).** $\int_0^1 \frac{1}{(x^2+1)^{3/2}} dx$. Here we use a trigonometric substitution. Let $x = \tan\theta \qquad dx = (\sec\theta)^2 d\theta$ We must also change the limits of integration: $x = 0 \to \theta = \arctan 0 = 0 x = 1 \to \theta = \arctan 1 = \frac{\pi}{4}$

$$\int_0^1 \frac{1}{(x^2+1)^{3/2}} dx = \int_0^{\pi/4} \frac{1}{((\tan\theta)^2+1)^{3/2}} ((\sec\theta)^2 d\theta) = \int_0^{\pi/4} \frac{(\sec\theta)^2}{((\sec\theta)^2)^{3/2}} d\theta$$

$$= \int_0^{\pi/4} \frac{1}{\sec\theta} d\theta = \int_0^{\pi/4} \cos\theta d\theta = \sin\theta \Big|_0^{\pi/4} = \boxed{\frac{\sqrt{2}}{2}}$$

**1.(d).** $\int \cos(\ln x) dx$ [Hint: Use integration by parts.] $u = \cos(\ln x) \qquad dv = 1dx$ $du = -\frac{\sin \ln x}{x} dx \qquad v = x$ $\int \cos(\ln x) dx = x\cos(\ln x) + \int \sin(\ln x) dx$. Now, I'll use integration by parts again, on the second term above: $u = \sin(\ln x) \qquad dv = 1dx$ $du = \frac{\cos \ln x}{x} dx \qquad v = x$ so, $\int \sin(\ln x) dx = x\sin(\ln x) - \int \cos(\ln x) dx$. Putting this all together, we have $\int \cos(\ln x) dx = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x) dx$ Hence, $2\int \cos(\ln x) dx = x\cos(\ln x) + x\sin(\ln x)$, So $\int \cos(\ln x) dx$

$$= \boxed{\frac{1}{2} x\cos(\ln x) + \frac{1}{2} x\sin(\ln x) + C}$$

**1.(e).** $\int 2\sin 2x \cdot (\sin x)^2 dx = \int 2 \cdot 2\sin x \cos x \cdot (\sin x)^2 dx = 4\int (\sin x)^3 \cos x dx$. Now do a $u$-substitution: $u = \sin x \qquad du = \cos x dx$ We have, $4\int u^3 du = 4 \cdot \frac{1}{4} u^4 + C$. Substituting back in we get $\boxed{(\sin x)^4 + C}$

**1.(f).** $\int \frac{5x-1}{(x-1)(x-2)}$ This is a partial fractions problem: $\frac{5x-1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \to 5x - 1 = A(x-2) + B(x-1) \to 5 = A + B \qquad -1 = -2A - B \to A = -4 \qquad B = 9$ Plugging back into the integral, we have: $\int \frac{5x-1}{(x-1)(x-2)} = -4\int \frac{1}{x-1} dx + 9\int \frac{1}{x-2} dx =$

$$\boxed{-4\ln|x-1| + 9\ln|x-2| + C}$$

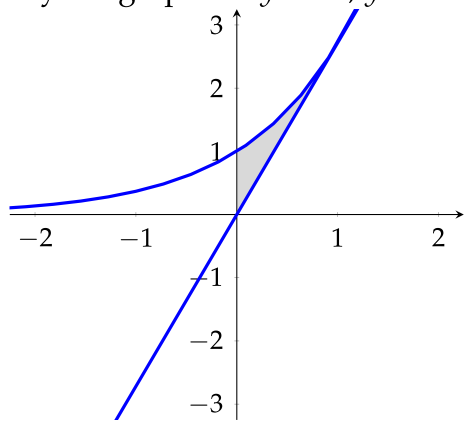
**1.(g).** $\int \sqrt{25-x^2}dx$. Here we use a trigonometric substitution: Let $x = 5\sin\theta \qquad dx = 5\cos\theta d\theta \to \int \sqrt{25-x^2}dx = \int \sqrt{25-5^2(\sin\theta)^2}(5\cos\theta d\theta) = \int \sqrt{25(\cos\theta)^2}(5\cos\theta) = 25\int(\cos\theta)^2 dx = \frac{25}{2}\int 1+\cos 2\theta d\theta = \frac{25}{2}\theta + \frac{25}{4}\sin 2\theta = \frac{25}{2}\theta + \frac{25}{4}2\sin\theta\cos\theta$ Now, from the substitution choice, we have $\sin\theta = \frac{x}{5}$, hence, $\cos\theta = \sqrt{1-(\sin\theta)^2} = \sqrt{1-\frac{x^2}{25}}$, and, $\theta = \arcsin\frac{x}{5}$. Putting this all together, we get $\frac{25}{2}\theta + \frac{25}{4}2\sin\theta\cos\theta = \frac{25}{2}(\arcsin\frac{x}{5}) + \frac{25}{4}\cdot 2\cdot(\frac{x}{5})\cdot(\frac{\sqrt{25-x^2}}{5}) + C =$ $\boxed{\frac{25}{2}(\arcsin\frac{x}{5}) + \frac{1}{2}x\sqrt{25-x^2}) + C}$

**1.(h).** $\int \frac{2x^2+x-1}{(x-3)(x^2+1)}$. This is another partial fractions problem. $\int \frac{2x^2+x-1}{(x-3)(x^2+1)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1} \to 2x^2+x-1 = A(x^2+1) + (Bx+C)(x-3) \to 2 = A+B \qquad 1 = -3B + C \qquad -1 = A-3C \to A = 2, B = 0, C = 1 \to \int \frac{2x^2+x-1}{(x-3)(x^2+1)} = 2\int \frac{1}{x-3}dx + \int \frac{1}{x^2+1}dx =$

$\boxed{2\ln|x-3| + \arctan x + C}$

| 2 |
| --- |
| 10 points |

**2.** Let $R$ be the region bounded by the graphs of $y = e^x$, $y = ex$ and the $y-$axis.




Set up but do not compute an integral expression for the following:

**2.(a).** The area of $R$.

**2.(b).** The volume of the solid of revolution obtained by rotating $R$ about the $x$-axis.

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SOLUTION

**2.(a).** $$\int_0^1 (e^x - ex)dx$$

**2.(b).** $$\pi \int_0^1 [(e^x)^2 - (ex)^2]dx$$

| 3 |
| --- |
| 10 points |

**3.** Determine if the integral $\int_1^\infty \frac{x}{x^2+4}dx$ converges or diverges by evaluating the integral.

SOLUTION

Let $u = x^2 + 4$, so that $du = 2xdx$. When $x = 1$, we have $u = 5$, and when $x \to \infty$, we have $u \to \infty$. It follows that

$$\int_1^\infty \frac{x}{x^2+4}dx = \frac{1}{2}\int_5^\infty \frac{1}{u}du = \frac{1}{2}\lim_{b\to\infty}\int_5^b \frac{1}{u}du = \frac{1}{2}\lim_{b\to\infty}\left(\ln u\Big|_5^b\right) = \frac{1}{2}[\lim_{b\to\infty}(\ln b - \ln 5)]$$

As $b \to \infty$, we have $\ln b \to \infty$, so the limit on the right diverges, and hence this integral

**diverges.**

**4.** Determine if the integral $\int_0^\infty \frac{1+\sin x}{e^x} dx$ converges or diverges.

| 4 |
|---|
| 10 points |

SOLUTION

The fact that $-1 \leq \sin x \leq 1$ implies that $0 = \frac{1-1}{e^x} \leq \frac{1+\sin x}{e^x} \leq \frac{1+1}{e^x} = \frac{2}{e^x}$. Now

$$\int_0^\infty \frac{2}{e^x} dx = \lim_{b\to\infty} \int_0^b \frac{2}{e^x} dx = \lim_{b\to\infty} -2e^{-x}\Big|_0^b = \lim_{b\to\infty} -2e^{-b} + 2 = 2,$$

so that by the Comparison Theorem, the integral $\int_0^\infty \frac{1+\sin x}{e^x} dx$ **converges.**