MATH 2300: Calculus II, Fall 2014
FINAL EXAM

Thursday, December 18, 2014

YOUR NAME:

Circle Your CORRECT Section

◯ **001** M. PELFREY .... (9AM)	◯ **006** S. WEINELL .... (3PM)
◯ **002** E. ANGEL .... (10AM)	◯ **007** C. BLAKESTAD .... (8AM)
◯ **003** E. ANGEL .... (11AM)	◯ **008** P. WASHABAUGH .... (1PM)
◯ **004** J. HARPER .... (12PM)	◯ **009** J. HARPER .... (3PM)
◯ **005** B. CHHAY .... (2PM)	◯ **010** K. PARKER .... (4PM)

Important note: SHOW ALL WORK. BOX YOUR ANSWERS. Calculators are not allowed. No books, notes, etc. Throughout this exam, please provide exact answers where possible. That is: if the answer is 1/2, do not write 0.499 or something of that sort; if the answer is π, do not write 3.14159.

Problem	Points	Score
1	8	
2	5	
3	7	
4	12	
5	10	
6	6	
7	7	
8	16	
9	9	
10	7	
11	4	
12	9	
TOTAL	100	

"On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work."

SIGNATURE:

1. (4 points each) Compute

(a) $\frac{\partial}{\partial x}\left(xe^{\sqrt{xy}}\right) = \frac{\partial}{\partial x}(x) \cdot e^{\sqrt{xy}} + x \frac{\partial}{\partial x}\left(e^{\sqrt{xy}}\right)$

$= e^{\sqrt{xy}} + x e^{\sqrt{xy}} \cdot \frac{\partial}{\partial x}\left(\sqrt{xy}\right)$

$= e^{\sqrt{xy}} + x e^{\sqrt{xy}} (xy)^{-1/2} \cdot y$

$= \underline{e^{\sqrt{xy}} \cdot \left(1 + \sqrt{xy}\right)}$

(b) $f_y(1, \pi)$ if $f(x, y) = 4x^2y + e^x + y\sin(xy)$

$f_y(1,\pi) = \frac{d}{dy} f(1,y)\Big|_{y=\pi}$

$= \frac{d}{dy}\left(4(1)^2 y + e^1 + y\sin(1 \cdot y)\right)\Big|_{y=\pi}$

$= \left(4 + y\cos(y) + \sin(y)\right)\Big|_{y=\pi}$

$= \underline{4 + \pi}$

2. (5 points) Write out the first three nonzero terms of the Taylor series for xe^{x^2} near $x = 0$.

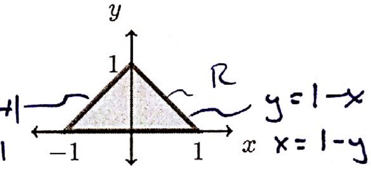
if $u = x^2$, $e^{x^2} = e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots + \frac{u^n}{n!} + \cdots$

$= 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots + \frac{(x^2)^n}{n!} + \cdots$

$xe^{x^2} = x\left(1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots\right)$

$\approx x + x^3 + \frac{x^5}{2!}$

3. (7 points) Find the total mass of the region shown below with density $\delta(x, y) = 1 + x + y$.



Integration wrt. dx first makes sense:

$M = \int_R \delta(x,y)\,dA = \int_{y=0}^{1}\int_{x=y-1}^{1-y} \delta(x,y)\,dx\,dy$

$= \int_{y=0}^{1}\int_{x=y-1}^{1-y} (1+x+y)\,dx\,dy$

$= \int_{y=0}^{1} \left[x + \frac{x^2}{2} + xy\right]_{x=y-1}^{1-y} dy$

$= \int_{y=0}^{1} \left(\left[(1-y) + \frac{(1-y)^2}{2} + (1-y)y\right] - \left[(y-1) + \frac{(y-1)^2}{2} + (y-1)y\right]\right) dy$

$= \int_{y=0}^{1} \left(2(1-y) + 2(1-y)y\right) dy = 2\int_{y=0}^{1} (1-y^2)\,dy$

$= 2\left[y - \frac{y^3}{3}\right]_{y=0}^{1} = 2\left[1 - \frac{1}{3}\right] = \frac{4}{3}$

4. (2 points each) Do the following converge or diverge? **CIRCLE** your answer, no work necessary.

a) $\sum_{n=1}^{\infty} \frac{(n+1)^2}{2n+3}$ CONVERGE **DIVERGE**

Div. Test

b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ CONVERGE **DIVERGE**

p-series $(p=\frac{1}{2})$

c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n+1}$ **CONVERGE** DIVERGE

AST.

d) $\sum_{n=1}^{\infty} \frac{5n+2}{n^2-7n}$ CONVERGE **DIVERGE**

Limit Comp. Test (to $\sum \frac{1}{n}$)

e) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ **CONVERGE** DIVERGE

Ratio Test: $L = \lim_n \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2} \right|$

$= \lim_n \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1.$

f) $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n$ **CONVERGE** DIVERGE

Root Test: $\lim_n \sqrt[n]{\left(\frac{1}{n}\right)^n} = \lim_n \frac{1}{n} = 0 < 1$

5. (a) (3 points) Write the Taylor series for $\sin(x)$ near $x = 0$.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(b) (5 points) Using the Lagrange error bound, what is the minimum degree Taylor polynomial of $\sin x$ near $x = 0$ necessary to approximate $\sin(0.1)$ to within $\frac{1}{5000}$?

n = degree.

if $n = 1$: $E_1(0.1) \leq \frac{M \cdot |0.1|^2}{2!}$ where $M \geq |-\sin x|$, for $x \in [0, 0.1]$.

$M = 1$ is a decent choice but we may even choose $M = 0.1 > \sin(0.1)$. Note $\sin x$ is increasing on

for $M = 0.1$: $E_1(0.1) \leq \frac{(0.1) \cdot |0.1|^2}{2!} = \frac{1}{2000}$

So we need a greater degree than $n=1$:

if $n=2$: $E_2(0.1) \leq \frac{M\,|0.1|^3}{3!}$ where $M \geq |-\cos x|$, for $x \in [0, 0.1]$

$M=1$ is the best bound, as $\cos(1) = 1$ and $\cos x$ is decreasing on $[0, 0.1]$. So

$$E_2(0.1) \leq \frac{1 \cdot |0.1|^3}{3!} = \frac{1}{6000}.$$

Thus degree 2 = degree 1 is the minimum degree, i.e. $P_2(x) = P_1(x)$.

(c) (2 points) Using the degree you found in part (b), approximate $\sin(0.1)$ to within $\frac{1}{5000}$.

$$\sin x \approx x = P_1(x).$$

$$\sin(0.1) \approx P_1(0.1) = \underline{0.1}.$$

6. (2 points each) Decide (without calculation) whether the integrals are positive, negative, or zero. Let D be the region inside the unit circle centered at the origin, let R be the right half of D, and let B be the bottom half of D. **CIRCLE** your answer.

a) $\int_D (y + y^5)\, dA$ POSITIVE NEGATIVE ZERO

b) $\int_R (y + y^5)\, dA$ POSITIVE NEGATIVE ZERO

c) $\int_B (y + y^5)\, dA$ POSITIVE NEGATIVE ZERO

7. (a) (4 points) Find the general solution to the differential equation $\frac{dy}{dx} = xy$.

Sep. of Var's: $\frac{dy}{y} = x\,dx \Rightarrow \int \frac{dy}{y} = \int x\,dx$

$\Rightarrow \ln|y| = \frac{x^2}{2} + C$

$\Rightarrow |y| = e^{x^2/2} \cdot e^{C}$ Let $A = e^C$, Note $A > 0$.

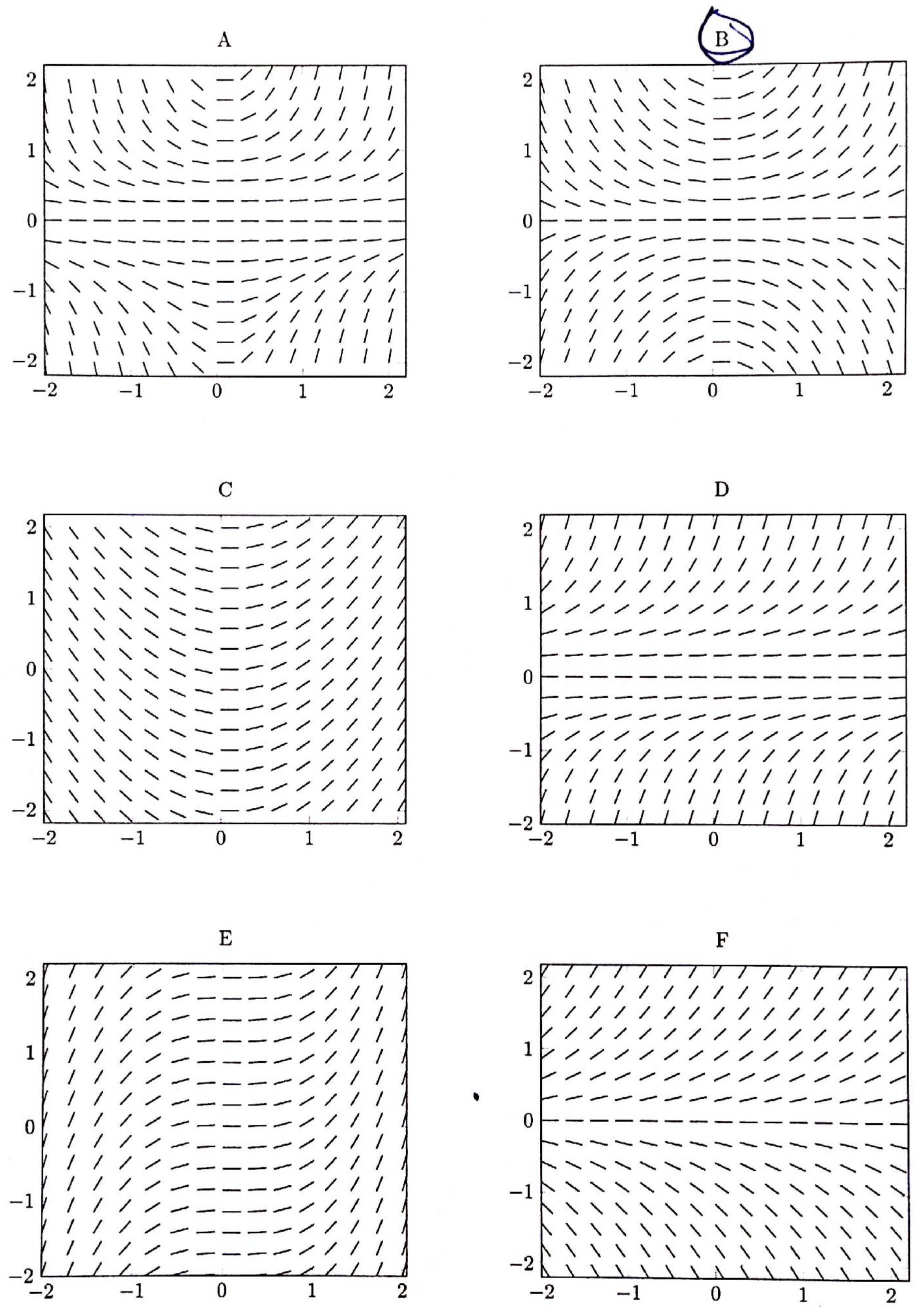
$\Rightarrow |y| = A e^{x^2/2}$

$\Rightarrow y = \pm A e^{x^2/2}$ Let $B = \pm A$. Note: $B \neq 0$

$\Rightarrow \underline{y = Be^{x^2/2}}$.

B is nonzero. In the first step we divided by y, losing the y=0 sol'n of the original eqn. Thus B can be chosen to be any real number.

(b) (3 points) Match the differential equation to one of the slope fields below. **CIRCLE** your choice.



8. (4 points each) Compute the following indefinite integrals.

(a) $\int \frac{\cos(\sqrt{y})}{\sqrt{y}}\, dy$ $= \int \cos(u)\, du$

$= \sin(u) + C$

$= \sin(\sqrt{y}) + C$

$u = \sqrt{y}$

$du = \frac{1}{2\sqrt{y}}\, dy$

$2du = \frac{1}{\sqrt{y}}\, dy$

(b) $\int \frac{1}{x^2-1}\, dx$ A sec sub is possible but we will use

Partial Fractions: $\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$

$\Rightarrow 1 = A(x+1) + B(x-1)$

if $x=1$: $1 = 2A \Rightarrow A = 1/2$

if $x=-1$: $1 = -2B \Rightarrow B = -1/2$. Thus

$\int \frac{1}{x^2-1}\, dx = \int \left(\frac{1/2}{x-1} + \frac{-1/2}{x+1}\right) dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$

(c) $\int \frac{x+1}{\sqrt{x}}\, dx$ $= \int (x^{1/2} + x^{-1/2})\, dx$

$= \frac{2x^{3/2}}{3} + 2x^{1/2} + C.$

Integration by parts is also possible, as is u-sub.

Using integration by parts twice:

(d) $\int \underbrace{\sin(x)}_{u_1}\underbrace{e^x\,dx}_{dv_1}$ $= u_1 v_1 - \int v_1\, du_1 \quad + C$

$= \sin(x) e^x - \int \underbrace{e^x}_{dv_2} \underbrace{\cos(x)}_{u_2} dx + C$

$= \sin(x)e^x - \left[u_2 v_2 - \int v_2\, du_2\right] + C$

$= \sin(x)e^x - e^x\cos(x) + \int e^x(-\sin(x))dx + C$

$u_1 = \sin x$
$dv_1 = e^x\, dx$
$du_1 = \cos x\, dx$
$v_1 = e^x$

$u_2 = \cos x$
$dv_2 = e^x dx$
$du_2 = -\sin x$
$v_2 = e^x$

$\Rightarrow \int \sin(x) e^x dx = \sin(x)e^x - e^x\cos(x) - \int e^x \sin(x)\, dx + C$

$\Rightarrow 2\int \sin(x)\, e^x = e^x(\sin(x) - \cos(x)) + C$

$\Rightarrow \int e^x \sin(x)\, dx = \underline{e^x(\sin(x) - \cos(x)) + C}$

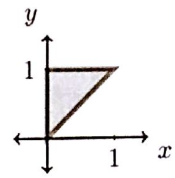
9. (3 points each) By setting one variable constant, find the equation of a plane that intersects the graph of $z = (x^2+1)\sin y + xy^2$ in a:

(a) parabola: $y = \pi/2$: $z = (x^2+1) + x(\pi)^2$. In fact, any y s.t. $y \neq n\pi$ for any $n \in \mathbb{N}$ works.

(b) straight line: $y = 0$. In fact, any y s.t. $y = n\pi$ for any $n \in \mathbb{N}$ works. gives $z = 0$ (in xz-plane).

(c) sine curve: $x = 0$. Gives: $z = \sin y$.

10. (7 points) Integrate e^{y^2} over the region shown below.



As $\int e^{y^2}$ has no antideriv. in terms of elementary fcns. wrt. y, integrate wrt. dx first.

$$\int_R e^{y^2}\,dA = \int_{y=0}^{1}\int_{x=0}^{y} e^{y^2}\,dx\,dy$$

$$= \int_{y=0}^{1}\left[x e^{y^2}\right]_{x=0}^{y} dy$$

$$= \int_{y=0}^{1} y e^{y^2}\,dy$$

$$= \int_{u=0}^{1} \frac{1}{2} e^{u}\,du$$

$$= \frac{1}{2} e^{u}\Big|_{u=0}^{1} = \frac{1}{2}(e-1)$$

Let $u = y^2$.
$du = 2y\,dy$
$\Rightarrow \frac{1}{2}du = y\,dy$
Also, $y=1 \Rightarrow u=1$
$y=0 \Rightarrow u=0$

11. (4 points) Which of the following integrals equals the area to the right of the line $x = 1$ and inside the circle $r = 2$? **CIRCLE** your choice.

(a) Area $= \frac{1}{2}\int_{-\pi/6}^{\pi/6} (2^2 - \cos^2\theta)\,d\theta$

(b) Area $= \frac{1}{2}\int_{0}^{\pi/6} (2^2 - \cos^2\theta)\,d\theta$

(c) Area $= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2^2 - \cos^2\theta)\,d\theta$

(d) Area $= \frac{1}{2}\int_{0}^{\pi/3} (2^2 - \cos^2\theta)\,d\theta$

(e) Area $= \frac{1}{2}\int_{-\pi/6}^{\pi/6} (2^2 - \sec^2\theta)\,d\theta$

(f) Area $= \frac{1}{2}\int_{0}^{\pi/6} (2^2 - \sec^2\theta)\,d\theta$

(g) Area $= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2^2 - \sec^2\theta)\,d\theta$

(h) Area $= \frac{1}{2}\int_{0}^{\pi/3} (2^2 - \sec^2\theta)\,d\theta$

Note: $x=1 \Leftrightarrow r\cos\theta = 1 \Leftrightarrow r = \frac{1}{\cos\theta} = \sec\theta.$

Also, $2 = \sec\theta$ has sol'ns $\pm\frac{\pi}{3}$.

12. (a) (3 points) Write the Taylor series of $\cos(x)$ near $x = 0$.

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

(b) (6 points) Does $\int_0^1 \frac{\cos(x)}{x}\, dx$ converge or diverge? Show your work.

$$\int_0^1 \frac{\cos(x)}{x}\, dx = \lim_{b \to 0^+} \int_b^1 \frac{\cos(x)}{x}\, dx$$

$$= \lim_{b \to 0^+} \int_b^1 \frac{1}{x} \cdot \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \right) dx$$

$$= \lim_{b \to 0^+} \int_b^1 \left(\frac{1}{x} - \frac{x}{2!} + \frac{x^3}{4!} - \cdots + (-1)^n \frac{x^{2n-1}}{(2n)!} + \cdots \right) dx$$

$$= \lim_{b \to 0^+} \left[\ln|x| - \frac{x^2}{2 \cdot 2!} + \frac{x^4}{4 \cdot 4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)(2n)!} \right]_b^1$$

$$= \lim_{b \to 0^+} \left[\left(\ln(1) - \frac{1}{2 \cdot 2!} + \frac{1}{4 \cdot 4!} - \cdots + (-1) \frac{1}{(2n)(2n)!} + \cdots \right) - \left(\ln|b| - \frac{b^2}{2 \cdot 2!} + \frac{b^4}{4 \cdot 4!} - \cdots + (-1)^n \frac{b^{2n}}{(2n)(2n)!} \cdots \right) \right]$$

$$= -\infty \quad \text{(Diverges)}$$

Because the terms in the first row converge, (AST or Ratio)

All the terms in the second row converge to 0, except $\ln|b|$ which diverges.