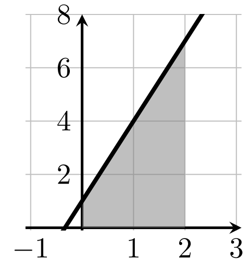
Calculating a definite integral from the limit of a Riemann Sum

Example: Evaluate $\int_0^2 3x + 1\,dx$ using the limit of right Riemann Sums.

This integral corresponds to the area of the shaded region shown to the right.

(Note: From geometry, this area is 8. So in this example, we already know the answer by another method)



- Slice it into n rectangles:

$$\text{Width of rectangles } \Delta x = \frac{2}{n}$$

- Right-hand Endpoints (x-coordinates):

$$\frac{2}{n}, \frac{4}{n}, \frac{6}{n}, ..., \frac{2n}{n}$$

(These are given by the formula $x_i = \frac{2i}{n}$)

- Height of Rectangles: plug the right-hand x-coordinates into $f(x) = 3x + 1$.

$$3\left(\frac{2}{n}\right) + 1, 3\left(\frac{4}{n}\right) + 1, 3\left(\frac{6}{n}\right) + 1, ..., 3\left(\frac{2n}{n}\right) + 1$$

(These heights are given by the $f(x_i) = 3\left(\frac{2i}{n}\right) + 1$)

- Area of Rectangles: height × width

$$\left(3\left(\frac{2}{n}\right)+1\right)\left(\frac{2}{n}\right), \left(3\left(\frac{4}{n}\right)+1\right)\left(\frac{2}{n}\right), ..., \left(3\left(\frac{2n}{n}\right)+1\right)\left(\frac{2}{n}\right)$$

(These areas are given by the formula $f(x_i)\Delta x = \left(3\left(\frac{2i}{n}\right)+1\right)\left(\frac{2}{n}\right)$)

- Riemann Sum (total area, sum of areas of all the rectangles):

$$\left(3\left(\frac{2}{n}\right)+1\right)\left(\frac{2}{n}\right) + \left(3\left(\frac{4}{n}\right)+1\right)\left(\frac{2}{n}\right) + ... + \left(3\left(\frac{2n}{n}\right)+1\right)\left(\frac{2}{n}\right)$$

(In summation notation: $\sum_{i=1}^{n}\left(3\left(\frac{2i}{n}\right)+1\right)\left(\frac{2}{n}\right)$)

- Expand and simplify the Riemann sum:

$$\begin{aligned}
&\left(3\left(\frac{2}{n}\right)+1\right)\left(\frac{2}{n}\right)+\left(3\left(\frac{4}{n}\right)+1\right)\left(\frac{2}{n}\right)+...+\left(3\left(\frac{2n}{n}\right)+1\right)\left(\frac{2}{n}\right)\\
&=\left(\frac{6}{n}+1\right)\left(\frac{2}{n}\right)+\left(\frac{12}{n}+1\right)\left(\frac{2}{n}\right)+...+\left(\frac{6n}{n}+1\right)\left(\frac{2}{n}\right)\\
&=\left(\frac{6}{n}\cdot\frac{2}{n}+\frac{2}{n}\right)+\left(\frac{12}{n}\cdot\frac{2}{n}+\frac{2}{n}\right)+...+\left(\frac{6n}{n}\cdot\frac{2}{n}+\frac{2}{n}\right)\\
&=\left(\frac{12}{n^2}+\frac{24}{n^2}+...+\frac{12n}{n^2}\right)+\left(\frac{2}{n}+\frac{2}{n}+...+\frac{2}{n}\right)\\
&=\frac{12}{n^2}\left(1+2+...+n\right)+n\cdot\frac{2}{n}\qquad\left(\text{Note that } 1+2+...+n=\frac{n(n+1)}{2}\right)\\
&=\frac{12}{n^2}\cdot\frac{n(n+1)}{2}+2=\frac{12}{2}\cdot\frac{n^2+n}{n^2}+2\\
&=6\left(1+\frac{1}{n}\right)+2\\
&=8+\frac{6}{n}
\end{aligned}$$

Or, the entire expansion and simplification can be done in sigma notation:

$$\begin{aligned}
\sum_{i=1}^{n}\left(3\left(\frac{2i}{n}\right)+1\right)\left(\frac{2}{n}\right)&=\sum_{i=1}^{n}\left(\frac{6i}{n}+1\right)\left(\frac{2}{n}\right)\\
&=\sum_{i=1}^{n}\left(\frac{12i}{n^2}+\frac{2}{n}\right)=\sum_{i=1}^{n}\frac{12i}{n^2}+\sum_{i=1}^{n}\frac{2}{n}\\
&=\frac{12}{n^2}\sum_{i=1}^{n}i+\frac{2}{n}\sum_{i=1}^{n}1\\
&=\frac{12}{n^2}\cdot\frac{n(n+1)}{2}+\frac{2}{n}\cdot n\qquad\left(\text{Note: }\sum_{i=1}^{n}i=\frac{n(n+1)}{2}\text{ and }\sum_{i=1}^{n}1=n\right)\\
&=6\cdot\frac{n^2+n}{n^2}+2=6\left(1+\frac{1}{n}\right)+2\\
&=8+\frac{6}{n}
\end{aligned}$$

- Take the limit of the Rieman Sums:

$$\lim_{n\to\infty}8+\frac{6}{n}=8$$

Therefore,

$$\int_0^2 3x+1\,dx=8$$