Section 7.1 - Integration by Substitution

We recall that integration "reverses" the derivative. From the calculation:

$$\frac{d}{dx}[e^{x^2}] = 2xe^{x^2},$$

we may conclude that

$$\int 2xe^{x^2} = e^{x^2}.$$

Up until this point, we have developed our rules for integration based on basic rules for the derivative. We now move towards "reversing" a more complicated derivative rule.

Goal: We want to "reverse" the chain rule.

Recall: (The Chain Rule): $\dfrac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$.

So we are going to develop a process to calculate the antiderivative $\int f'(g(x)) \cdot g'(x)dx$.

The Method of Substitution: We calculate $\int f'(g(x)) \cdot g'(x)dx$ as follows:

Step 1: Set $u = g(x)$.

Step 2: Calculate $du = g'(x)dx$.

Step 3: Plug in u and du: $\quad \displaystyle\int f'(g(x)) \cdot g'(x)dx = \int f'(u)du$

Step 4: Calculate the integral: $\quad = f(u) + C$.

Step 5: Plug your x's back in: $\quad = f(g(x)) + C$.

Note: The book uses w instead of u, both letters are commonly used. We have opted to use u for these notes.

Example 1: Calculate $\displaystyle\int x^2e^{x^3}dx$.

Calculation:

$$\begin{aligned}
\int x^2e^{x^3}dx &= \int e^{x^3}(x^2dx) \\
&= \int e^u\left(\frac{1}{3}du\right) \\
&= \frac{1}{3}\int e^u du \\
&= \tfrac{1}{3}e^u + C \\
&= \tfrac{1}{3}e^{x^3} + C
\end{aligned}$$

Side work:

$$\begin{aligned}
u &= x^3 \\
du &= 3x^2dx \\
\tfrac{1}{3}du &= x^2dx
\end{aligned}$$

Good Choices for u:
· Things you cannot integrate ($\ln x$, $\sin^{-1} x$, $\tan^{-1} x$).
· Insides of functions.
· Denominator of a fraction.
· "Something" when you see something and its derivative.

Example 2: Calculate $\displaystyle\int \frac{1}{x \ln x} dx$.
We see there is a $\ln x$, so we try choosing $u = \ln x$.

$$\begin{aligned} \int \frac{1}{x\ln x}dx &= \int \frac{1}{\ln x}\frac{1}{x}dx & u &= \ln x \\ &= \int \frac{1}{u}du & du &= \tfrac{1}{x}dx \\ &= \ln|u| + C \\ &= \ln|\ln x| + C \end{aligned}$$

Example 3: Calculate $\displaystyle\int \tan x dx$.
We first observe that $\tan x = \frac{\sin x}{\cos x}$. This allows us to see "something", and its derivative. We choose $\cos x$ for u since $\cos x$ is in the denominator.

$$\begin{aligned} \int \tan x dx &= \int \frac{\sin x}{\cos x}dx & u &= \cos x \\ &= -\int \frac{1}{\cos x} \cdot -\sin x dx & du &= -\sin x dx \\ &= -\int \frac{1}{u}du \\ &= -\ln|\cos x| + C \\ &= \ln(|\cos x|^{-1}) + C \\ &= \ln|\sec x| + C \end{aligned}$$

Using Substitution with Definite Integrals

Method 1: Use substitution to find the antiderivative with x in it, then evaluate.

Example 4: Calculate $\displaystyle\int_0^1 x^2 e^{x^3} dx$.
From Example 1, we know that $\displaystyle\int x^2 e^{x^3} = \frac{1}{3}e^{x^3} + C$. So $\frac{1}{3}e^{x^3}$ is an antiderivative of $x^2e^{x^3}$. Thus

$$\int_0^1 x^2 e^{x^3} dx = \frac{1}{3}e^{x^3}\Big|_0^1 = \frac{1}{3}e^{1^3} - \frac{1}{3}e^{0^3} = \frac{e-1}{3}.$$

Method 2: Switch the bounds of your integral with the substitution.

Example 5: Calculate $\int_1^3 x\sqrt{x^2-1}dx$.

Calculation:

$$\begin{aligned}
\int_1^3 x\sqrt{x^2-1}dx &= \frac{1}{2}\int_1^3 (x^2-1)^{1/2}\cdot 2xdx \\
&= \frac{1}{2}\int_0^8 u^{1/2}du \\
&= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]\Big|_0^8 \\
&= \frac{1}{3}[8^{3/2}-0^{3/2}] \\
&= \frac{8^{3/2}}{3}
\end{aligned}$$

Side work:

$u = x^2 - 1$

$du = 2xdx$

at $x = 1, u = 1^2 - 1 = 0$

at $x = 3, u = 3^2 - 1 = 8$