Section 6.4: Second Fundamental Theorem of Calculus

Let $f(x)$ be a function defined on an interval I. Suppose we want to find an antiderivative $F(x)$ of $f(x)$ on the interval I. Sometimes, we are able to find an expression for $F(x)$ analytically. For example, if $f(x) = x^2$, then we can take $F(x) = \frac{x^3}{3}$. However, there are elementary functions $f(x)$ (functions that are combinations of constants, powers of $x, \sin x, \cos x, e^x$, and $\ln x$) that do not have an antiderivative $F(x)$ that can be expressed as an elementary function. One such example of an elementary function that does not have an elementary antiderivative is $f(x) = \sin(x^2)$.

The Second Fundamental Theorem of Calculus studied in this section provides us with a tool to construct antiderivatives of continuous functions, even when the function does not have an elementary antiderivative:

Second Fundamental Theorem of Calculus. *Let f be a continuous function defined on an interval I. Fix a point a in I and define a function F on I by*

$$F(x) = \int_a^x f(t)dt.$$

Then F is an antiderivative of f on the interval I, i.e. $F'(x) = f(x)$ on I.

A proof of the Second Fundamental Theorem of Calculus is given on pages 318–319 of the textbook.

We note that $F(x) = \int_a^x f(t)dt$ means that F is the function such that, for each x in the interval I, the value of $F(x)$ is equal to the value of the integral $\int_a^x f(t)dt$. Furthermore, $F(a) = \int_a^a f(t)dt = 0$, and so F is the anitderivative of f that satisfies $F(a) = 0$.

Now since $\int_a^x f(t)dt$ is an antiderivative of $f(x)$, then the general form of an antiderivative of $f(x)$ is given by

$$F(x) = C + \int_a^x f(t)dt,$$

where C is a constant. In this case, we compute

$$F(a) = C + \int_a^a f(t)dt = C + 0 = C.$$

Therefore we have the result that the general form of an antiderivative of $f(x)$ is given by

$$F(x) = C + \int_a^x f(t)dt, \text{ where } C = F(a).$$

We also note that the fact that $\int_a^x f(t)dt$ is an antiderivative of $f(x)$ in the Second Fundamental Theorem of Calculus can be expressed as

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Example 1. Let $f(x) = \sin(x^2)$. Then the function

$$F(x) = \int_0^x \sin(t^2)dt$$

is the antiderivative of f that satisfies $F(0) = 0$. For every real number x, we can find the value of $F(x)$ by computing numerically the integral $\int_0^x \sin(t^2)dt$. We give a few values of F in the table below:

x	-3	-2	-1	0	1	2	3
$F(x)$	-0.7736	-0.8048	-0.3103	0	0.3103	0.8048	0.7736

We remark that the table suggests that F is an odd function, i.e. $F(-x) = -F(x)$. Indeed, since the function $f(t) = \sin(t^2)$ is even (as $f(-t) = \sin((-t)^2) = \sin(t^2) = f(t)$), we must have, following a result from Section 5.4, that

$$\int_{-x}^0 \sin(t^2)dt = \int_0^x \sin(t^2)dt.$$

From here,

$$F(-x) = \int_0^{-x} \sin(t^2)dt = -\int_{-x}^0 \sin(t^2)dt = -\int_0^x \sin(t^2)dt = -F(x),$$

and so F is an odd function.

Example 2. Suppose we know that the function $f(x)$ is such that $f'(x) = e^{-x^2}$ and $f(0) = 2$. Then an expression for $f(x)$ is given by

$$f(x) = 2 + \int_0^x e^{-t^2}dt.$$

If we are asked to find the value of $f(3)$, we have

$$f(3) = 2 + \int_0^3 e^{-t^2}dt = 2 + 0.8862 = 2.8862,$$

where the above integral is computed numerically.

Example 3.

(a) Find $\dfrac{d}{dx}\displaystyle\int_2^x \ln(t^2+1)dt$.

A direct application of the Second Fundamental Theorem of Calculus yields

$$\frac{d}{dx}\int_2^x \ln(t^2+1)dt = \ln(x^2+1).$$

(b) Find $\dfrac{d}{dt}\displaystyle\int_t^\pi \cos(z^3)dz$.

First, we need to switch the limits of integration, and then we apply the Second Fundamental Theorem of Calculus:

$$\frac{d}{dt}\int_t^\pi \cos(z^3)dz = \frac{d}{dt}\left(-\int_\pi^t \cos(z^3)dz\right) = -\frac{d}{dt}\int_\pi^t \cos(z^3)dz = -\cos(t^3).$$

(c) Find $\dfrac{d}{dx}\displaystyle\int_2^{x^3} \sin(t^2)dt$.

Let $G(x) = \displaystyle\int_2^{x} \sin(t^2)dt$. Then, by the Second Fundamental Theorem of Calculus,

$$G'(x) = \frac{d}{dx}\int_2^{x} \sin(t^2)dt = \sin(x^2).$$

Since $\displaystyle\int_2^{x^3} \sin(t^2)dt = G(x^3)$, we are asked to find $\dfrac{d}{dx}\Big(G(x^3)\Big)$. By the chain rule,

$$\frac{d}{dx}\Big(G(x^3)\Big) = 3x^2G'(x^3).$$

Hence

$$\frac{d}{dx}\int_2^{x^3} \sin(t^2)dt = \frac{d}{dx}\Big(G(x^3)\Big) = 3x^2G'(x^3) = 3x^2\sin((x^3)^2) = 3x^2\sin(x^6).$$

(d) Find $\dfrac{d}{dt}\displaystyle\int_{t^2}^{\cos t} \sqrt{1+x^4}dx$.

We start by breaking up the integral in two and then switching the limits of integration in the first integral:

$$\int_{t^2}^{\cos t} \sqrt{1+x^4}dx = \int_{t^2}^{0} \sqrt{1+x^4}dx + \int_{0}^{\cos t} \sqrt{1+x^4}dx = -\int_{0}^{t^2} \sqrt{1+x^4}dx + \int_{0}^{\cos t} \sqrt{1+x^4}dx.$$

Let $G(t) = \displaystyle\int_0^{t} \sqrt{1+x^4}dx$. Then we can write

$$\int_{t^2}^{\cos t} \sqrt{1+x^4}dx = -\int_{0}^{t^2} \sqrt{1+x^4}dx + \int_{0}^{\cos t} \sqrt{1+x^4}dx = -G(t^2) + G(\cos t).$$

By the chain rule,

$$\frac{d}{dt}\int_{t^2}^{\cos t} \sqrt{1+x^4}dx = \frac{d}{dt}\Big(-G(t^2) + G(\cos t)\Big) = -2tG'(t^2) - (\sin t)G'(\cos t).$$

By the Second Fundamental Theorem of Calculus, we have

$$G'(t) = \frac{d}{dt}\int_0^{t} \sqrt{1+x^4}dx = \sqrt{1+t^4}.$$

Hence

$$\frac{d}{dt}\int_{t^2}^{\cos t} \sqrt{1+x^4}dx = -2tG'(t^2) - (\sin t)G'(\cos t)$$

$$= -2t\sqrt{1+(t^2)^4} - (\sin t)\sqrt{1+(\cos t)^4} = -2t\sqrt{1+t^8} - (\sin t)\sqrt{1+(\cos t)^4}.$$