Section 6.3 - Differential Equations

Definition: A *differential equation* is an equation involving derivatives of an unknown function.

Definition: Finding the *general solution* to the differential equation $\frac{dy}{dx} = f(x)$, means finding the general antiderivative $y = F(x) + C$ with $F'(x) = f(x)$.

Example 1: Find the general solution of

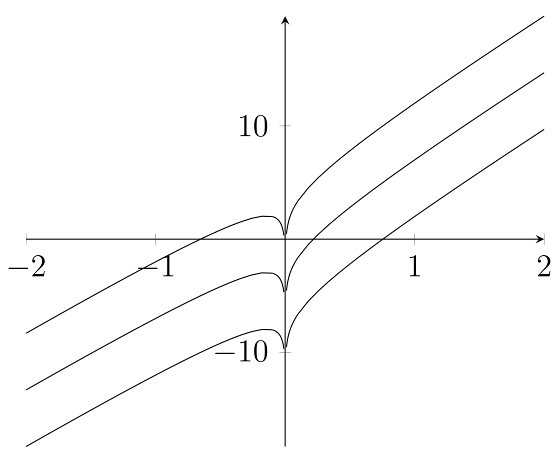
$$\frac{dy}{dx} = \frac{1}{x} + 7.$$

We recall from the rules of Section 6.2, that one antiderivative of $f(x) = \frac{1}{x} + 7$ is $F(x) = \ln|x| + 7x$.
So our general solution is

$$y = \ln|x| + 7x + C$$

where C is any constant.

From our general solution, we can plot specific solutions to this differential equation. Below we have plotted solutions for $C = -5, 0, 5$.



Definition: An *initial value problem* is a differential equation together with a coordinate:

$$\frac{dy}{dx} = f(x), \qquad y(x_0) = y_0.$$

An initial value problem usually has only one solution (you figure out what the "C" is).

Example 2: Find the solution of the initial value problem

$$\frac{dy}{dx} = \sin x, \qquad y(0) = 1.$$

Our general solution is

$$y = -\cos x + C.$$

To find our particular solution, we plug in our point and solve for C:

$$\begin{aligned} y &= -\cos x + C \\ 1 &= -\cos(0) + C \\ 1 &= -1 + C \\ 2 &= C \end{aligned}$$

Now $y = -\cos x + 2$ is our unique solution this initial value problem.

Fact: An object moving under the influence of gravity has constant acceleration g, where the value of g is approximately

$$g = -9.8 \text{ m/sec}^2, \qquad \text{or} \qquad g = -32 \text{ ft/sec}^2.$$

Note that our acceleration is negative since we normally assign a coordinate system to our problem where up is thought of as positive.

Recall: The derivative of position is velocity, and the derivative of velocity is acceleration.

Example 3: A man throws a ball straight up. He releases the ball from a height of 2 meters with a speed of 10 m/sec. Find, as functions of time, its position and velocity. When does the ball hit the ground, and how fast is it going at that time?

From the fact that $a = -9.8$ and our initial speed is 10 m/sec, and assuming the ball was released at time $t = 0$, we get the initial value problem

$$\frac{dv}{dt} = -9.8, \qquad v(0) = 10.$$

We can solve this initial value problem to get our velocity function.

$$\begin{aligned} v &= -9.8t + C \\ 10 &= -9.8(0) + C \\ 10 &= C \\ v &= -9.8t + 10 \end{aligned}$$

From this velocity function and the fact that the ball was 2 meters up when released, we get the initial value problem

$$\frac{ds}{dt} = -9.8t + 10, \qquad s(0) = 2.$$

We solve this to get our position function.

$$\begin{aligned} s &= -9.8(\tfrac{1}{2}t^2) + 10t + K \\ s &= -4.9t^2 + 10t + K \\ 2 &= -4.9(0)^2 + 10(0) + K \\ 2 &= K \\ s &= -4.9t^2 + 10t + 2 \end{aligned}$$

So we have found our functions for position and velocity. To find when the ball hit the ground, we plug in $s = 0$ to our position function and solve for t.

$$\begin{aligned} 0 &= -4.9t^2 + 10t + 2 \\ t &= \frac{-10 \pm \sqrt{10^2 - 4(-4.9)(2)}}{2(-4.9)} \\ t &= -0.1835, 2.2243 \end{aligned}$$

We dismiss the negative answer since the ball hit the ground after it was released, not before. We conclude the ball hit the ground 2.2243 seconds after being released.
To get the impact velocity, we find the velocity at the moment the ball hits, which we found to be $t = 2.2243$ seconds. So we substitute $t = 2.2243$ into the velocity function

$$v = -9.8(2.2243) + 10 = -11.7983.$$

We conclude the ball was heading down (because of the negative) at a speed of 11.7983 m/sec.