MATH 1300

Section 6.2 - Constructing Antiderivatives Analytically

Goal: Given a formula for $f(x)$, we want to find a formula for $F(x)$ such that $F'(x) = f(x)$ – such an $F(x)$ is known as the *antiderivative.*

We begin with the most basic case, the identically zero function. We would expect such a function to have a horizontal tangent line everywhere, meaning it is impossible for the function to be increasing or decreasing at any point on its domain. This gives the following result:

Propostion. *The Antiderivative of the Zero Function.*

If $F'(x) = 0$ on the interval (a, b), then $F(x) = C$ on $[a, b]$, for some constant C.

Proof. This is a reformulation of the Constant Function Theorem from section 3.10. □

We also notice that if $F(x)$ is an antiderivative of $f(x)$, then so is $F(x)+C$ for any constant C – after all, the derivative of a constant is identically zero, so the derivative of either formula will still give $f(x)$. We can express this idea more formally in the following way:

Propostion. *If F, G are both antiderivatives of f on an interval, then $F(x) = G(x) + C$, where C is a constant.*

Proof. Since $F' = f$ and $G' = f$, we have $F' = G'$. Thus, $F' - G' = 0$, and therefore $(F - G)' = 0$ by the difference rule for derivatives. But then $F - G = C$, so $F(x) = G(x) + C$. □

Because all antiderivatives of $f(x)$ are of the form $F(x)+C$, it makes sense to come up with notational shorthand for this family of functions. As the antiderivative is closely tied to the notion of integration, we adopt notation very similar to that of the integral itself.

Definition. The **indefinite integral** of a function, f, is denoted $\int f(x)\,dx$. Furthermore, $\int f(x)\,dx = F(x) + C$, where F is an antiderivative of f.

WARNING: Do NOT confuse $\int_a^b f(x)\,dx$ and $\int f(x)\,dx$. The first is a number, whereas the latter is a *family of functions.* Complicating matters, the word "integration" is frequently used interchangeably for both the process of taking a definite integral *and* for the process of finding an antiderivative. Context will usually clarify which meaning is intended.

Using this new notation, we can more quickly describe the antiderivatives of entire families of functions. But first, a quick motivating example:

Example 1. Find $\int 5\,dx$.

Solution. We know that a constant value is left after differentiating a linear formula. This would suggest we want an equation of a line, $5x$. But since we want to denote the whole family of functions whose derivative is 5, we write

$$\int 5\,dx = 5x + C,$$

where C is any constant.

Check: $\frac{d}{dx}(5x + C) = 5 + 0 = 5$, as desired.

This suggests the following:

Propostion. *The Antiderivative of a Constant.*

If k is a constant, then

$$\int k\,dx = kx + C,$$

where C is any constant.

Proof. If k is a constant, then

$$\frac{d}{dx}(kx + C) = k\frac{d}{dx}(x) + 0 = k. \qquad \square$$

In fact, all of our old differentiation rules have corresponding antidifferentiation rules – if we can recognize the function as a derivative, we automatically know its antiderivative.

For all of the following, C is an arbitrary constant.

Propostion. *The Reverse Power Rule.*

If $n \neq -1$, then

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Proof. If $n \neq -1$, then $n + 1 \neq 0$. So

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = \frac{1}{n+1}\frac{d}{dx}(x^{n+1}) + 0 = \frac{1}{n+1} \cdot (n+1)x^n = x^n. \qquad \square$$

Propostion. *The Antiderivative of $\frac{1}{x}$.*

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

Proof. If $x > 0$, then $\ln|x| + C = \ln(x) + C$, and

$$\frac{d}{dx}(\ln(x) + C) = \frac{1}{x} + 0 = \frac{1}{x}.$$

If $x < 0$, then $\ln|x| + C = \ln(-x) + C$, and

$$\frac{d}{dx}(\ln(-x) + C) = -1 \cdot \frac{1}{-x} + 0 = \frac{1}{x}. \qquad \square$$

Propostion. *The Antiderivative of* e^x.

$$\int e^x \, dx = e^x + C.$$

Proof.

$$\frac{d}{dx}\left(e^x + C\right) = e^x + 0 = e^x. \qquad \square$$

Propostion. *The Antiderivatives of* $\sin(x)$ *and* $\cos(x)$.

$$\int \sin(x) \, dx = -\cos(x) + C.$$

$$\int \cos(x) \, dx = \sin(x) + C.$$

Proof.

$$\frac{d}{dx}\left(-\cos(x) + C\right) = -\frac{d}{dx}(\cos(x)) + 0 = -(-\sin(x)) = \sin(x),$$

and

$$\frac{d}{dx}(\sin(x) + C) = \cos(x) + 0 = \cos(x). \qquad \square$$

Following techniques similar to those in the proofs of the preceding propositions, it is possible to construct antidifferentiation rules for a wide array of functions. We shall look at a more specific example.

Example 2. Find $\displaystyle\int (7x^2 + 4x^3) \, dx$.

Solution. We know that $x^3/3$ is an antiderivative of x^2, and that $x^4/4$ is an antiderivative of x^3. We therefore expect a result along the lines of

$$\int (7x^2 + 4x^3) \, dx = 7\left(\frac{x^3}{3}\right) + 4\left(\frac{x^4}{4}\right) + C = \frac{7}{3}x^3 + x^4 + C.$$

Check: $\displaystyle\frac{d}{dx}\left(\frac{7}{3}x^3 + x^4 + C\right) = \frac{7}{3} \cdot 3x^2 + 4x^3 + 0 = 7x^2 + 4x^3$, as desired.

Actually, this example motivates our next theorem, which will allow us to make better use of the many preceding propositions.

Theorem. *Properties of Antiderivatives: Sum and Constant Multiples.*

In indefinite integral notation, where f, g are continuous functions and c is a constant,

1. $\displaystyle\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx.$
2. $\displaystyle\int cf(x)\, dx = c\int f(x)\, dx.$

Proof. The results are immediate consequences of the sum and constant multiple rules for derivatives. □

Example 3. Find $\displaystyle\int (\cos(x) + 4e^x)\, dx.$

Solution. We break the antiderivative into two terms:

$$\int (\cos(x) + 4e^x)\, dx = \int \cos(x)\, dx + \int 4e^x\, dx = \int \cos(x)\, dx + 4\int e^x\, dx = \sin(x) + 4e^x + C.$$

Check: $\dfrac{d}{dx}(\sin(x) + 4e^x + C) = \cos(x) + 4\dfrac{d}{dx}(e^x) + 0 = \cos(x) + 4e^x$, as desired.

As we saw in section 5.3, the Fundamental Theorem of Calculus gives us a way to calculate definite integrals from their antiderivatives. Indeed, if we denote $F(b) - F(a)$ by $F(x)|_a^b$, the theorem says that if $F' = f$ and f is continuous, then

$$\int_a^b f(x)\, dx = F(x)|_a^b = F(b) - F(a).$$

This means that if we can find $F(x)$, we can compute the value of $\displaystyle\int_a^b f(x)\, dx$, and our antidifferentiation techniques have given us the machinery to do just that.

Example 4. Compute $\displaystyle\int_1^2 (4x^3 + 2x)\, dx$ using the Fundamental Theorem of Calculus.

Solution. Since $F(x) = x^4 + x^2$ is an antiderivative of $f(x) = 4x^3 + 2x$ by the reverse power rule,

$$\int_1^2 (4x^3 + 2x)\, dx = F(x)|_1^2 = F(2) - F(1),$$

gives

$$\int_1^2 (4x^3 + 2x)\, dx = \left[x^4 + x^2\right]_1^2 = 2^4 + 2^2 - (1^4 + 1^2) = 18.$$

Notice that in this example we used the antiderivative $x^4 + x^2$, but $x^4 + x^2 + C$ will work for any constant C, since the C will just cancel out in the final computation. In fact, when you are evaluating definite integrals, it suffices to always choose the $C = 0$ case.

Example 5. Compute $\int_0^{\frac{\pi}{4}} \sec(x)\tan(x)\,dx$ exactly.

Solution. We use the Fundamental Theorem. Since $F(x) = \sec(x)$ is an antiderivative of $f(x) = \sec(x)\tan(x)$,

$$\int_0^{\frac{\pi}{4}} \sec(x)\tan(x)\,dx = [\sec(x)]_0^{\frac{\pi}{4}} = \sqrt{2} - 1.$$