MATH 1300 **Lecture Notes** **Wednesday, November 20, 2013**

Section 6.1: Antiderivatives Graphically and Numerically

Let $f(x)$ be a function defined on an interval I. We say that $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$ for all x in I. We note that if $F(x)$ is an antiderivative of $f(x)$, then, since $\frac{d}{dx}(F(x) + C) = F'(x) = f(x)$ for any constant C, the function $F(x) + C$ is an antiderivative of $f(x)$ for any constant C. The functions of the form $F(x) + C$ form a family of antiderivatives for the function $f(x)$.

Example 1. Let $f(x) = 3x^2$. We note that $\frac{d}{dx}(x^3) = 3x^2$, and, in general, $\frac{d}{dx}(x^3 + C) = 3x^2$ for any constant C. Hence, for any constant C, the function $F(x) = x^3 + C$ is an antiderivative of $f(x) = 3x^2$.

Suppose now that we are asked to find an antiderivative $F(x)$ of $f(x) = 3x^2$ that is such that $F(2) = 9$. Using $F(x) = x^3 + C$ and $F(2) = 9$, we get $2^3 + C = 9$, and so $C = 1$. Therefore our antiderivative is $F(x) = x^3 + 1$.

If we are given the graph of the derivative $f'(x)$ of a function $f(x)$, we can sketch a possible graph of $f(x)$. We need to keep in mind the following:

(a) If $f'(x) > 0$ on an interval I then f is increasing on I, and if $f'(x) < 0$ on an interval I then f is decreasing on I.

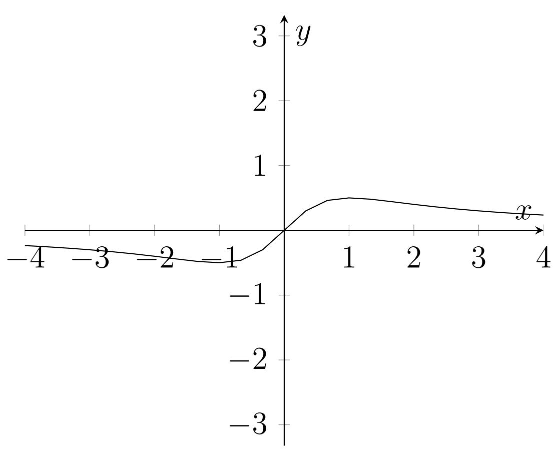
(b) Points $x = c$ such that $f'(c) = 0$ are points where f has a horizontal tangent line. Furthermore, if f' changes sign at $x = c$ from positive to negative then f has a local maximum at $x = c$, and if f' changes sign at $x = c$ from negative to positve then f has a local minimum at $x = c$.

(c) If f' is increasing on an interval I then f is concave up on I, and if f' is decreasing on an interval I then f is concave down on I.

(d) Local extrema (i.e. local maxima and local minima) of f' correspond to inflection points of f.

Once we sketch a possible graph of $f(x)$, then any vertical shift of the graph will give us another possibility for $f(x)$, since a vertical shift corresponds to adding a constant to the function

Example 2. The graph of $f'(x)$ is shown below:



From the graph, we notice that:

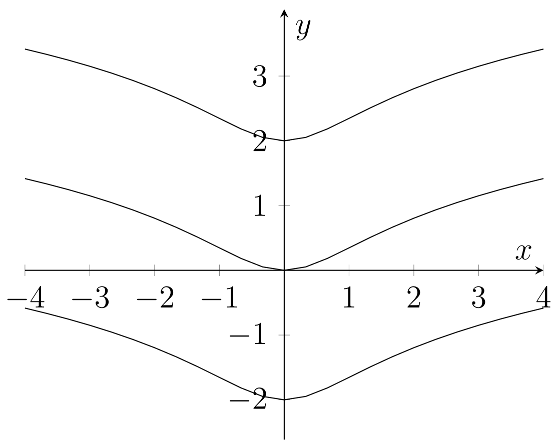
(a) $f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$. Hence f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

(b) $f'(0) = 0$ and f' changes sign from negative to positive at $x = 0$. Hence f has a local minimum at $x = 0$.

(c) f' is decreasing on $(-\infty, -1)$ and $(1, \infty)$, and f' is increasing on $(-1, 1)$. Hence f is concave down on $(-\infty, -1)$ and $(1, \infty)$, and f is concave up on $(-1, 1)$.

(d) f' has a local minimum at $x = -1$ and a local maximum at $x = 1$. Hence f has inflection points at $x = \pm 1$.

Three graphs of $f(x)$, one with $f(0) = -2$, one with $f(0) = 0$, and one with $f(0) = 2$ are given below:



If f' is continuous, the Fundamental Theorem of Calculus states that

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Therefore, if we know the value of $f(a)$, we can compute $f(b)$ from the formula

$$f(b) = f(a) + \int_a^b f'(x)dx.$$

Example 3. Suppose $f'(x) = x^2 e^x$ and $f(1) = 6$. Compute $f(3)$.

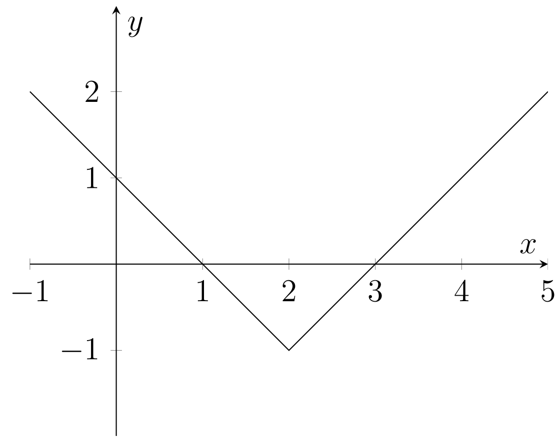
Applying the formula from above,

$$f(3) = f(1) + \int_1^3 f'(x)dx = f(1) + \int_1^3 x^2 e^x dx.$$

We compute $\int_1^3 x^2 e^x dx$ numerically to get $\int_1^3 x^2 e^x dx = 97.7094$. Hence we conclude that

$$f(3) = f(1) + \int_1^3 x^2 e^x dx = 6 + 97.7094 = 103.7094.$$

Example 4. Compute the values of f at all of its local extrema and inflection points, and then sketch the graph of f, given that $f(0) = 1$ and that the graph of $f'(x)$ is shown below:



Looking at the graph of f', we start with the same analysis as in Example 2:

(a) $f'(x) > 0$ on $(-\infty, 1)$ and $(3, \infty)$, and $f'(x) < 0$ on $(1, 3)$. Hence f is increasing on $(-\infty, 1)$ and $(3, \infty)$, and decreasing on $(1, 3)$.

(b) $f'(1) = 0$ and f' changes sign from positive to negative at $x = 1$. Also, $f'(3) = 0$ and f' changes sign from negative to positive at $x = 3$. Hence f has a local maximum at $x = 1$ and a local minimum at $x = 3$.

(c) f' is decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$. Hence f is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$.

(d) f' has a local minimum at $x = 2$. Hence f has an inflection point at $x = 2$.

Since f has local extrema at $x = 1, 3$ and an inflection point at $x = 2$, we need to compute $f(1)$, $f(2)$, and $f(3)$.

Using the fact that $f(0) = 1$, we compute $f(1)$ from the formula

$$f(1) = f(0) + \int_0^1 f'(x)dx.$$

We can compute $\int_0^1 f'(x)dx$ using geometry to obtain

$$f(1) = f(0) + \int_0^1 f'(x)dx = 1 + \frac{(1)(1)}{2} = 1 + \frac{1}{2} = \frac{3}{2}.$$

Similarly, we compute

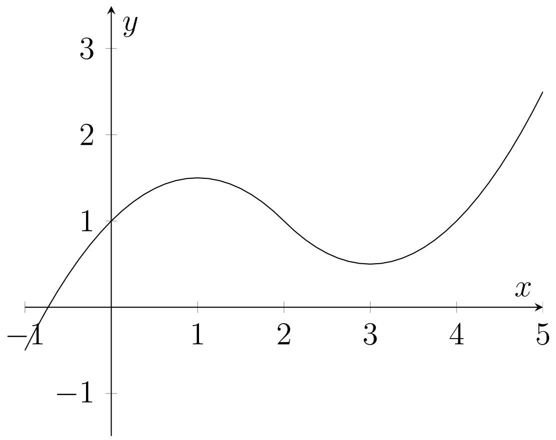
$$f(2) = f(1) + \int_1^2 f'(x)dx = \frac{3}{2} - \frac{(1)(1)}{2} = \frac{3}{2} - \frac{1}{2} = 1$$

and

$$f(3) = f(2) + \int_2^3 f'(x)dx = 1 - \frac{(1)(1)}{2} = 1 - \frac{1}{2} = \frac{1}{2}.$$

(Note that when we compute the integrals, we add areas above the x-axis and subtract areas below the x-axis.)

Using of all the above information, the graph of $f(x)$ is given below:



y
3
2
1
x
−1
1
2
3
4
5
−1