MATH 1300

Section 5.4 - Theorems about Definite Integrals

Up to this point in our dealings with definite integrals, $\int_a^b f(x)\,dx$, we have only considered the case where $a < b$. Ideally, we would like to be able to handle cases where $a \geq b$, since this will make our notation more general; and indeed, we can do so. Additionally, we find that sometimes it is possible to combine two integrals, or to break a single integral up into two parts.

Theorem. *Properties of Limits of Integration.*

If a, b, c are any real numbers, and f is a continuous function, then

1. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$
2. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

Proof. 1. From the definition of the definite integral, we have

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i) \cdot \frac{b-a}{n} \right),$$

as well as

$$\int_b^a f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i) \cdot \frac{a-b}{n} \right).$$

So we are free to write

$$\begin{aligned}
\int_a^b f(x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i) \cdot \frac{b-a}{n} \right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i) \cdot \frac{-(a-b)}{n} \right) \\
&= \lim_{n\to\infty} \left[-\sum_{i=1}^{n} \left(f(x_i) \cdot \frac{a-b}{n} \right) \right] \\
&= -\lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i) \cdot \frac{a-b}{n} \right) \\
&= -\int_b^a f(x)\,dx,
\end{aligned}$$

as desired.

2. Suppose $a < c < b$. Then the area under the curve from a to c, plus the area under the curve from c to b, will be the area under the curve from a to b by simple geometry (See Fig. 1). This is precisely the desired equation

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

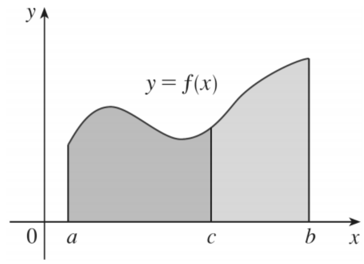

Fig. 1 *A graphical illustration of the $a < c < b$ case.*

All other cases, such as $a < b < c$, follow from similar geometric arguments in conjunction with the first part of this theorem.

□

Example 1. Given that $\int_0^1 (4x^3 + 2x)\,dx = 2$ and $\int_0^2 (4x^3 + 2x)\,dx = 20$, what is the value of $\int_1^2 (4x^3 + 2x)\,dx$?

Solution. By the additive property,

$$\int_1^2 (4x^3 + 2x)\,dx = \int_1^0 (4x^3 + 2x)\,dx + \int_0^2 (4x^3 + 2x)\,dx.$$

By the reversal of limits property,

$$\int_1^0 (4x^3 + 2x)\,dx = -\int_0^1 (4x^3 + 2x)\,dx.$$

So

$$\begin{aligned}
\int_1^2 (4x^3 + 2x)\,dx &= \int_1^0 (4x^3 + 2x)\,dx + \int_0^2 (4x^3 + 2x)\,dx \\
&= -\int_0^1 (4x^3 + 2x)\,dx + \int_0^2 (4x^3 + 2x)\,dx \\
&= -(2) + 20 \\
&= 18.
\end{aligned}$$

Now that we have introduced properties that allow us to freely manipulate our limits of integration, we naturally become curious as to what techniques we might be able to utilize to manipulate the integrands themselves, in order to make the actual integrals easier to compute. These techniques will occupy our attention for the remainder of this section.

Theorem. *Properties of Sums and Constant Multiples of the Integrand.*

Let f, g be continuous functions, and let c be a constant.

1. $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$
2. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$

Proof. 1. From the definition of the definite integral, we have

$$\int_a^b (f(x) \pm g(x))\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i) \pm g(x_i))\, \Delta x, \quad \Delta x = \frac{b-a}{n}.$$

Thus

$$\begin{aligned}
\int_a^b (f(x) \pm g(x))\ dx &= \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i) \pm g(x_i))\, \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i)\Delta x \pm g(x_i)\Delta x) \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i)\Delta x \pm \sum_{i=1}^{n} g(x_i)\Delta x \right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \pm \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\Delta x \\
&= \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx,
\end{aligned}$$

as desired.

2. From the definition of the definite integral, we have

$$\int_a^b cf(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} cf(x_i)\Delta x, \quad \Delta x = \frac{b-a}{n}.$$

Consequently,

$$\begin{aligned}
\int_a^b cf(x)\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} cf(x_i)\Delta x \\
&= \lim_{n\to\infty} c \sum_{i=1}^{n} f(x_i)\Delta x \\
&= c \cdot \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= c \int_a^b f(x)\, dx,
\end{aligned}$$

as desired.

□

Example 2. Evaluate the definite integral $\displaystyle\int_0^3 (3+6x)\, dx$ exactly.

Solution. Using the preceding theorem, we can break this integral up as follows:

$$\int_0^3 (3+6x)\, dx = \int_0^3 3\, dx + \int_0^3 6x\, dx = \int_0^3 3\, dx + 6\int_0^3 x\, dx.$$

We know graphically and from previous integral evaluations that $\displaystyle\int_0^3 3\, dx$ is precisely the area of a square with side length 3. Similarly, we know that $\int_0^3 x\, dx$ is the area of a triangle of base and height length 3. We conclude that

$$\int_0^3 3\, dx = 3 \cdot 3 = 9,$$

and that

$$\int_0^3 x\,dx = \frac{1}{2}\cdot 3\cdot 3 = \frac{9}{2}.$$

Thus

$$\int_0^3 (3+6x)\,dx = \int_0^3 3\,dx + 6\int_0^3 x\,dx = 9 + 6\left(\frac{9}{2}\right) = 36.$$

The preceding theorem has an immediate and straightforward consequence for the computation of the area *between* two curves, as well as the area of functions with symmetries.

Propostion. *Area Between Curves.*

If f, g are continuous functions and the graph of $f(x)$ lies above the graph of $g(x)$ for $a \le x \le b$, then the area between f and g for $a \le x \le b$ is given by $\int_a^b (f(x) - g(x))\,dx$.

Propostion. *Using Symmetry to Evaluate Integrals.*

If f is even, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$. If g is odd, then $\int_{-a}^a g(x)\,dx = 0$.

Example 3. Given that $\int_0^2 x^3\,dx = 4$, evaluate $\int_{-2}^2 x^3\,dx$ exactly.

Solution. x^3 is well-known to be an odd function. Consequently, by the preceding proposition,

$$\int_{-2}^2 x^3\,dx = 2\int_0^2 x^3\,dx = 2(4) = 8.$$

The proposition regarding the area between curves motivates our next technique – if, on the interval $a \le x \le b$, one can trap a function between two boxes, one can use the boxes to approximate the integral of the function, since the area beneath the curve must be bounded below by the smaller box and bounded above by the larger box. This is illustrated in the following image:

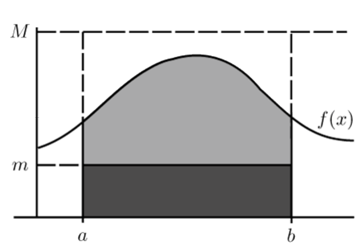

Fig. 2 *A function, $f(x)$, with $m \le f(x) \le M$ for all x in $[a, b]$.*

This idea can also be generalized to the notion of trapping by a larger function – after all, if $g(x)$ is always greater than $f(x)$ on $[a, b]$, certainly the area under the curve of g must exceed the area under the curve of f. Both of these ideas result in the following theorem:

Theorem. *Comparison of Definite Integrals.*

Let f, g be continuous functions.

1. *If $m \le f(x) \le M$ for all x in $[a, b]$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.*

2. If $f(x) \leq g(x)$ for all x in $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

Example 4. Explain why $0 \leq \displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\,dx \leq \frac{\pi}{2}$.

Solution. $\cos(x)$ is continuous and is bounded above and below by $-1 \leq \cos(x) \leq 1$ for all x – in particular, $0 \leq \cos(x) \leq 1$ on $[0, \frac{\pi}{2}]$. So by the preceding theorem,

$$0\left(\frac{\pi}{2} - 0\right) \leq \int_0^{\frac{\pi}{2}} \cos(x)\,dx \leq 1\left(\frac{\pi}{2} - 0\right).$$

Thus,

$$0 \leq \int_0^{\frac{\pi}{2}} \cos(x)\,dx \leq \frac{\pi}{2}.$$

Lastly, we note that, as mentioned in Section 5.3, the Fundamental Theorem of Calculus provides an exact way of computing certain definite integrals.

Example 5. Calculate $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\,dx$.

Solution. We know from our trigonometric derivatives that if $F(x) = \sin(x)$, then $F'(x) = \cos(x)$. So by the Fundamental Theorem of Calculus,

$$\int_0^{\frac{\pi}{2}} \cos(x)\,dx = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1.$$

This powerful technique, known as *antidifferentiation*, will be explored more fully in the next chapter.