1. **The Fundamental Theorem of Calculus**

Theorem: *If f is continuous on the interval $[a, b]$ and F is a function such that $F'(t) = f(t)$, then*

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

This theorem is so important that it is called the fundamental theorem of calculus. It is sometimes called the first fundamental theorem of calculus to distinguish it from the second fundamental theorem of calculus which will be discussed in section 6.4. The theorem provides a way of computing the exact values of definite integrals without having to find limits of Riemann sums.

The function F in the theorem is called the antiderivative of f. Now the problem of finding a definite integral of a function f is reduced to finding an antiderivate of f: a function whose derivative is f. Finding antiderivatives is not an easy business though: we have learned several rules to find derivatives of functions given by formulas, but now we have to find methods to reverse that process. In this course we will only learn a few simple ways to find antiderivatives. At the beginnig of Calculus II some more complicated methods will be discussed. However even then we won't be able to find antiderivatives of all functions given by arbitrary formulas: it is in fact proven that there exist elementary functions (functions given by formulas of $+, -, \cdot, /, \sin, \cos, \ln, e^x$, etc.) whose antiderivatives cannot be described by elementary formulas.

With a slight modification, the theorem can be put like this:

If F has a continuous derivative on the closed interval $[a, b]$ then

$$\int_a^b F'(t) = F(b) - F(a).$$

Thus, by the theorem, the operations of differentiation and integration are inverses to each other in a certain sense. This interpretation will be reinforced by the second fundamental theorem later.

This form of the theorem tells us that the definite integral of the rate of change gives the total change. For example, in the case of a one dimensional motion, the definite integral of the velocity between $t = t_1$ and $t = t_2$ gives the total change in the position between t_1 and t_2: $\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$, where $s(t)$ is the position of the object at time t and $v(t)$ is the velocity.

A precise proof of the theorem would belong to an Analysis course, but to make the theorem believable we have the following intuitive argument. Let's divide the interval $[a, b]$ into n subintervals of equal length, where n is some large enough number. Then using the Riemann sum approximation of the integral and the fact that on a small interval the derivative is close to the difference quotient we have:

$$\int_a^b F'(t)\,dt \approx \sum_{i=0}^{n-1} F'(t_i)\,\Delta t \approx \sum_{i=0}^{n-1} \frac{\Delta F}{\Delta t}\,\Delta t = \sum_{i=0}^{n-1} (F(t_{i+1}) - F(t_i)) =$$
$$= (F(t_1) - F(t_0)) + (F(t_2) - F(t_1)) + (F(t_3) - F(t_2)) + \ldots + (F(t_n) - F(t_{n-1})) =$$
$$= F(t_n) - F(t_0) = F(b) - F(a).$$

2. **Examples.**

(a) Find the exact value of the definite integral $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x\,dx$.

Solution. We know a function whose derivative is cosine: it is the sine function. Thus by the fundamental theorem of calculus we have:

$$\int_{-\pi/2}^{\pi/2} \cos x\,dx = \sin(\pi/2) - \sin(-\pi/2) = 1 - (-1) = 2.$$

(b) Find the area under graph of $f(x) = 3x^2$ between $x = 3$ and $x = 5$.

Solution. This area is given by the definite integral $\displaystyle\int_3^5 3x^2\,dx$. To find an antiderivative of $3x^2$, we have to reverse the power-rule: $3x^2 = (x^3)'$. Thus the area is $\displaystyle\int_3^5 3x^2\,dx = 5^3 - 3^3 = 125 - 27 = 98$.

Notice that the antiderivative of $f(x) = 3x^2$ is not unique. For example the function $F(x) = x^3 + 1$ is also an antiderivative of f. Of course, the result we get using this antiderivative is the same: $F(5) - F(3) = (5^3 + 1) - (3^3 + 1) = 5^3 - 3^3 = 98$.

3. **Units for the Definite Integral.**

To find units for the definite integral $\displaystyle\int_a^b f(x)\,dx$, we consider the definition of the integral which roughly says that it is the limit of the Riemann sums $\sum f(x_i)\,\Delta x$. In the Riemann sum the units of each term $f(x_i)\Delta x$ are units of $f(x)$ times units of x. Summing and taking limits will not change the units, therefore we have:

The units for the definite integral $\displaystyle\int_a^b f(x)\,dx$ *are "units of* $f(x)$ *times units of* x*", or in other words "units of the output of* f *times units of the input of* f*".*

For example, if time is measured in seconds and velocity is measured in meter/seconds, then the integral of the velocity has units meter/seconds times seconds, that is meters. This is what we expect because the integral of the velocity gives the distance travelled.

4. **The Average Value of a Function over an Interval.**

Suppose we wanted to find the average temperature at a given place over a 24 hour period from midnight to midnight. If we are lazy, we would measure the temperature four times during the day: at midnight, at 6am, at noon and at 6pm. Then we would take the average of the four values, thus average temperature $\approx (1/4)(f(0) + f(6) + f(12) + f(18))$, where $f(t)$ is the temperature at time t, t measured in hours since midnight. This would give a rough estimate of our idea of an average temperature. To improve the result we would have to measure the temperature more often, say every hour. Then we get: average temperature $\approx (1/24)(f(0)+f(1)+f(2)+\ldots+f(23)$. Of course, this can be improved as well by measuring the temperature every half an hour, every minute etc., to get more and more accurate values. As we increase the number of measurements, the results will tend to some value that we define to be the average temperature over the 24 hour period. (We have not had a definition for this up to now, we only had some idea of what the average temperature is.) An easy computation shows (see book, page 276) that the limit of the averages of the measurements as the number of measurements go to infinity will be the formula $\frac{1}{24}\int_0^{24} f(t)\,dt$. This can be generalized to any function to give the following

Definition. *The average value of f over the interval $[a, b]$ is defined to be*

$$Av_{[a,b]}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example. The value, V, of a Tiffany lamp, worth \$225 in 1975, increases at 15% per year. Its value in dollars t years after 1975 is given by $V = 225(1.15)^t$. Find the average value of the lamp over the period 1975-2010.

Solution. As 2010-1975 = 35, t runs from 0 to 35. Thus using the definition of the average value of a function over an interval we have that the average value of the lamp over the given period is

$$\frac{1}{35-0}\int_0^{35} 225(1.15)^t\,dt \approx \$6080.$$

To find the value of the integral we used WolframAlpha, although it would not have been hard to find an antiderivative of the integrand and use the first fundamental theorem of calculus.