Goal: We would like to find a method for determining the area between a general curve and the x-axis (we say this is the area underneath the curve). If the curve is the graph of a function that gives the velocity of an object with respect to time, then the area between the curve, the horizontal axis, and the vertical lines $t = t_1$ and $t = t_2$ will be the distance the object has traveled between times t_1 and t_2.

1 Sigma Notation

Sigma, or summation, notation is a compact way to write the sum of a large number of terms. Say we have n terms $t_1, t_2, \ldots, t_n$ (where the dots represent the $n-3$ terms between t_2 and t_n). Then their sum can be written as

$$t_1 + t_2 + \cdots + t_n = \sum_{i=1}^{n} t_i.$$

The "i" in sigma notation is called the *index*. The sum above is evaluated by letting i be every natural number satisfying $1 \leq i \leq n$ and adding all the resulting terms (the endpoints of this interval are determined by the numbers at the base and top of Σ). For example,

$$\sum_{i=1}^{5} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5) = 3 + 6 + 9 + 12 + 15 = 45$$

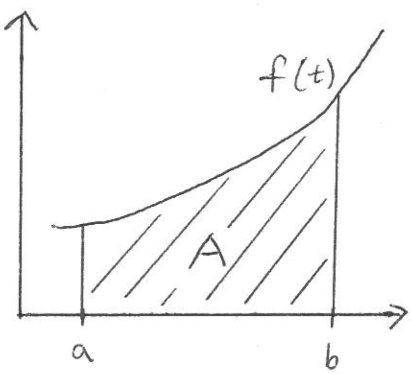
and

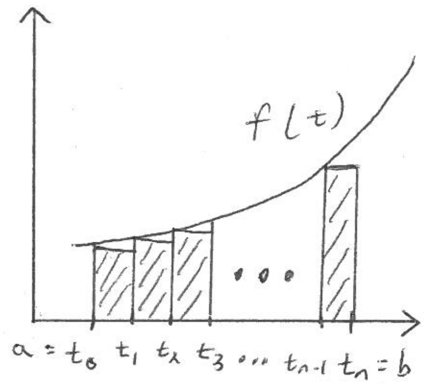
$$\sum_{i=3}^{6} i^2 = 3^2 + 4^2 + 5^2 + 6^2 = 9 + 16 + 25 + 36 = 86.$$

2 The Definite Integral

Suppose we are given a graph of a function, i.e. a curve, and asked to find the area A between this graph, the horizontal axis, and the vertical lines $t = a$ and $t = b$. Two of the ways we will estimate the area of A are with left-hand sums and right-hand sums. Both methods require us to subdivide the interval $[a, b]$ into n equal subdivisions. Looking at the figure below, we can see that each segment $[t_i, t_{i+1}]$ is of length $\Delta t = t_{i+1} - t_i = \dfrac{b-a}{n}$. Note that $a = t_0$ since it is the left-hand endpoint and $b = t_n$ since it is the right-hand endpoint.

The first way to estimate the area under the curve using rectangles is to take Δt as the width of each rectangle and the value of the function f at the left endpoint of each interval as the height of each rectangle as shown in the figure below.

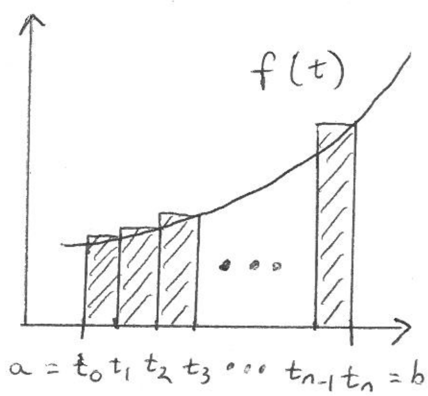




When we add up the areas of all these rectangles we get a **left-hand sum** (abbreviated LHS) since we are taking value of f at the left endpoint of each interval for the height of our rectangle. So

$$\begin{aligned}\text{Estimate of area A} = \text{Left-hand sum} &= f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_n)\Delta t \\ &= \sum_{i=0}^{n-1} f(t_i)\Delta t\end{aligned}$$

The second way to estimate the area under the curve is like the first in every way except that we take the right endpoint as the height of our rectangle.



When we add up the areas of all these rectangles we get a **right-hand sum** (abbreviated RHS) since we are taking the value of f at the right endpoint of each interval for the height of our rectangle. So

$$\begin{aligned}\text{Estimate of area } A = \text{Right-hand sum} &= f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t \\ &= \sum_{i=1}^{n} f(t_i)\Delta t\end{aligned}$$

Notice that we start at $i = 1$ and end at $i = n$ instead of $i = 0$ and $i = n-1$, respectively, since t_0 is not a right endpoint of any subdivision and t_n is the right-hand endpoint of the last subdivision.

Suppose f is continuous for $a \leq t \leq b$. We define the **definite integral** of f from a to b, written

$$\int_a^b f(t)\,dt,$$

to be the limit of the left-hand or right-hand sums (the limit is the same) with n subdivisions of $a \leq t \leq b$ as n goes to infinity. I.e.,

$$\int_a^b f(t)\,dt = \lim_{n\to\infty}(\text{Left-hand sum}) = \lim_{n\to\infty}\left(\sum_{i=0}^{n-1} f(t_i)\Delta t\right)$$

and

$$\int_a^b f(t)\,dt = \lim_{n\to\infty}(\text{Right-hand sum}) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(t_i)\Delta t\right)$$

Each of these sums is called a *Riemann sum*, f is called the *integrand*, and a and b are called the *limits of integration.*

Remark: It should be noted that the variable t in the definition of the definite integral is a dummy variable. It doesn't matter if you use a different value in place of t since the definite integral will have the same value. I.e., $\int_a^b f(t)\,dt = \int_a^b f(x)\,dx = \int_a^b f(\theta)\,d\theta = \ldots$.

Intuitively, as we increase the number of subdivisions of our interval $[a, b]$, the value of the left-hand sums and right-hand sums of a curve should give increasingly better approximations of the area under the curve. This is indeed the case and it turns out that as we let the number of subdivision go to infinity, the left-hand sums and right-hand sums converge to the actual area under the curve. In other words, When $f(t) \geq 0$ and $a < b$:

$$\text{Area under curve and above the horizontal axis between } a \text{ and } b = \int_a^b f(t)\,dt$$

When $f(t)$ is negative, then the value $f(t_i)\Delta t$ that previously represented the area of a rectangle is negative. If that is the case, the area below the horizontal axis is counted negatively in the calculation of a definite integral. In other words,

When $f(t)$ is positive for some t values and negative for others, and $a < b$, then

$$\int_a^b f(t)\,dt = (\text{area above horizontal axis}) - (\text{area below horizontal axis})$$

Example. (a) Calculate the left-hand and right-hand sums with $n = 4$ subdivsions to estimate the value of $\int_1^3 (3t+1)\,dt$.

(b) Use the limit of the left hand sum as $n \to \infty$ to calculate $\int_1^3 (3t+1)\,dt$ exactly.

Solution. (a) First we calculate $\Delta t = (b-a)/n = (3-1)/4 = 0.5$. The endpoints of our subdivisions are $t_0 = 1$, $t_1 = 1.5$, $t_2 = 2$, $t_3 = 2.5$, $t_4 = 3$. We have

$$\begin{aligned}
\text{LHS} = \sum_{i=0}^{3} f(t_i)\Delta t &= f(1)\cdot 0.5 + f(1.5)\cdot 0.5 + f(2)\cdot 0.5 + f(2.5)\cdot 0.5 \\
&= (3(1)+1)\cdot 0.5 + (3(1.5)+1)\cdot 0.5 + (3(2)+1)\cdot 0.5 + (3(2.5)+1)\cdot 0.5 \\
&= 12.5
\end{aligned}$$

$$\begin{aligned}
\text{RHS} = \sum_{i=1}^{4} f(t_i)\Delta t &= f(1.5)\cdot 0.5 + f(2)\cdot 0.5 + f(2.5)\cdot 0.5 + f(3)\cdot 0.5 \\
&= (3(1.5)+1)\cdot 0.5 + (3(2)+1)\cdot 0.5 + (3(2.5)+1)\cdot 0.5 + (3(3)+1)\cdot 0.5 \\
&= 15.5
\end{aligned}$$

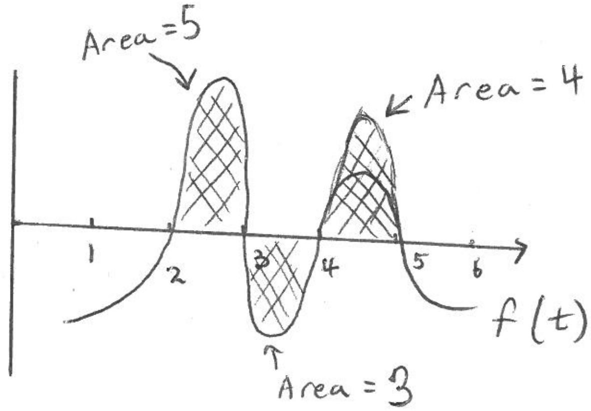
(b) Using the formula $\sum_{i=0}^{n-1} i = 0 + 1 + 2 + \cdots + n - 1 = \frac{n(n-1)}{2}$ and $\Delta t = (3-1)/n = 2/n$, we find

$$
\begin{aligned}
\int_1^3 (3t+1)\,dt &= \lim_{n\to\infty}\left(\sum_{i=0}^{n-1}(3t_i+1)\Delta t\right)\\
&= \lim_{n\to\infty}\left[(3t_0+1)\Delta t + (3t_1+1)\Delta t + \cdots + (3t_{n-1}+1)\Delta t\right]\\
&= \lim_{n\to\infty}\left[\left(3\left(1+0\cdot\left(\frac{2}{n}\right)\right)+1\right)\left(\frac{2}{n}\right)+\left(3\left(1+1\cdot\left(\frac{2}{n}\right)\right)+1\right)\left(\frac{2}{n}\right)+\right.\\
&\qquad\left.\cdots+\left(3\left(1+(n-1)\cdot\left(\frac{2}{n}\right)\right)+1\right)\left(\frac{2}{n}\right)\right]\\
&= \lim_{n\to\infty}\left(\sum_{i=0}^{n-1}\left(3\left(1+i\cdot\left(\frac{2}{n}\right)\right)+1\right)\left(\frac{2}{n}\right)\right)\\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1}\left(4+\frac{6i}{n}\right)\left(\frac{2}{n}\right)\\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1}\left(\frac{8}{n}+\frac{12i}{n^2}\right)\\
&= \lim_{n\to\infty}\left[\frac{8}{n}\sum_{i=0}^{n-1}1+\frac{12}{n^2}\sum_{i=0}^{n-1}i\right]\\
&= \lim_{n\to\infty}\left[\frac{8}{n}(n)+\frac{12}{n^2}\left(\frac{n(n-1)}{2}\right)\right]\\
&= \lim_{n\to\infty}\left[8+\frac{6n^2-6}{n^2}\right]\\
&= 8+6\\
&= 14
\end{aligned}
$$

You should notice that the actual value of the integral falls between the value of the LHS and the RHS. This is guaranteed to happen for monotonic functions.

□

Example. Calculate $\int_2^5 f(t)\,dt$ where $f(t)$ is given in the figure below.



Solution. We sum up the area above the horizontal axis and subtract the sum of the area below the horizontal axis.

$$\int_2^5 f(t)\,dt = 5 + 4 - 3 = 6$$

$\square$

In general, when a function f is increasing or decreasing the LHS and RHS give bounds on the actual value of the definite integral. One will be an upper bound and one will be a lower bound depending on whether f is increasing or decreasing. It's useful to know the value of the difference between the upper and lower bounds of the actual value of the integral. The closer together they are, the more accurate they have to be. For a function that is increasing or decreasing throughout the interval $[a, b]$, we have

$$|\text{Difference between upper and lower esitmates}| = |f(b) - f(a)| \cdot \Delta t$$

3 Measuring Distance Traveled

One application of the definite integral is calculating the distance an object travels between two times $t = a$ and $t = b$ given a graph of the object's velocity as a function of time. If the velocity curve $v(t)$ of an object with respect to time is positive for all values of t between a and b, then the area under the curve between time $t = a$ and time $t = b$, given by the definite integral $\int_a^b v(t)\,dt$, will be the distance the object traveled between a and b. If the velocity curve of an object is sometimes positive and sometimes negative, then the definite integral $\int_a^b v(t)\,dt$ represents the net change in postition between times a and b.

Example. Suppose a car is traveling at a constant speed of 5 mi/hr for 3 hours. To find the distance the car has traveled we can use the formula

$$\text{distance} = (\text{rate}) \cdot (\text{time})$$

to find that the distance the car has traveled is $(5 \text{ mi/hr})(3 \text{ hr}) = 15$ mi.

Now think of the car's speed as a function of time, i.e. $v(t) = 5$ mi/hr. If we graph this line between $t = 0$ and $t = 3$, we find that the area enclosed by the x-axis and the lines $v(t) = 5$, $t = 0$, and $t = 3$ is (height)(width) = (5mi/hr)(3hr) = 15mi. So the area of the rectangle corresponds exactly to the distance the car has traveled in 3 hours.

$\square$

Most of the time you won't be able to find the area underneath the curve using a geometric formula like we did in the example above. Instead, you can estimate the area underneath the curve using left-hand sums and right-sums.

Example: Suppose that the velocity of a car is given in mi/hr by the function $v(t) = t^2 + 1$ and we want to find the distance the car travels between $t = 1$ and $t = 7$. This function is represented in a table below

Time (hr)	1	2	3	4	5	6	7
Velocity (mi/hr)	2	5	10	17	26	37	50

To find a lower estimate for the distance traveled by the car, we assume that the car goes a constant speed during each hour determined by its speed at the beginning of the hour. Then

$$\text{Lower estimate for distance traveled } = \text{LHS} = 2 \cdot 1 + 5 \cdot 1 + 10 \cdot 1 + 17 \cdot 1 + 26 \cdot 1 + 37 \cdot 1 = 90 \text{ mi}$$

To find an upper estimate for the distance traveled by car, just assume that the car goes a constant speed during each hour determined by the speed at the end of each hour. Then

$$\text{Upper estimate for distance traveled } = \text{RHS} = 5 \cdot 1 + 10 \cdot 1 + 17 \cdot 1 + 26 \cdot 1 + 37 \cdot 1 + 50 \cdot 1 = 145 \text{ mi}$$

Therefore, the actual distance traveled by the car is somewhere between the lower estimate and the upper estimate.

$\square$

To get a more accurate estimate for the distance the car has traveled, we just take a velocity measurement more frequently, say every half an hour instead of every hour. Then our lower and upper estimates for the distance the car in the previous example has traveled should be more accurate. This is equivalent to taking a larger number of subdivisions to estimate the area under a curve using rectangles.