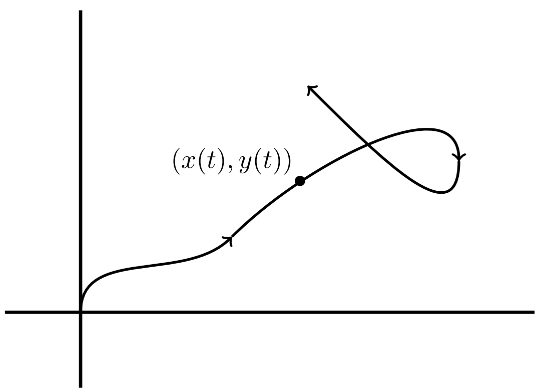
MATH 1300
SECTION 4.8: PARAMETRIC EQUATIONS

A **parametric equation** is a collection of equations

$$x = x(t)$$
$$y = y(t)$$

that gives the variables x and y as functions of a parameter t.

Any real number t then corresponds to a point in the xy-plane given by the coordinates $(x(t), y(t))$. If we think about the parameter t as time, then we can interpret $t = 0$ as the point where we start, and as t increases, the parametric equation traces out a curve in the plane, which we will call a **parametric curve**.



The result is that we can "draw" curves just like an Etch-a-sketch!

animated parametric circle from wolfram

Date: 11/06.

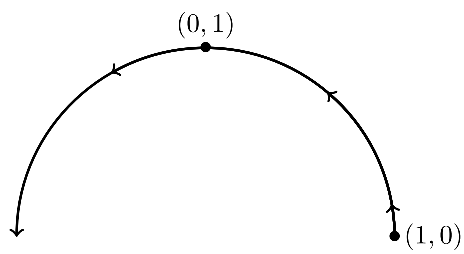
Let's consider an example. Suppose we have the following parametric equation:

$$x = \cos(t)$$
$$y = \sin(t)$$

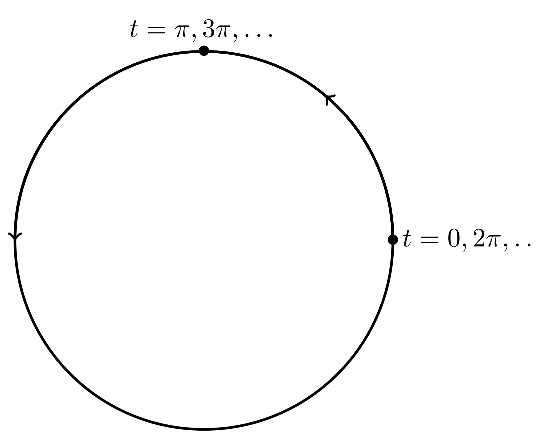
We know from the pythagorean theorem that this parametric equation satisfies the relation $\boxed{x^2 + y^2 = 1}$, so we see that as t varies over the real numbers we will trace out the unit circle!

Lets look at the curve that is drawn for $0 \leq t \leq \pi$. Just picking a few values we can observe that this parametric equation parametrizes the upper semi-circle in a counter clockwise direction.

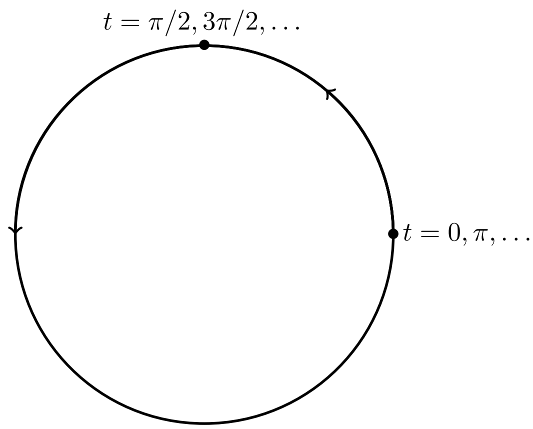
t	x(t)	y(t)
0	1	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	0	1
π	-1	0



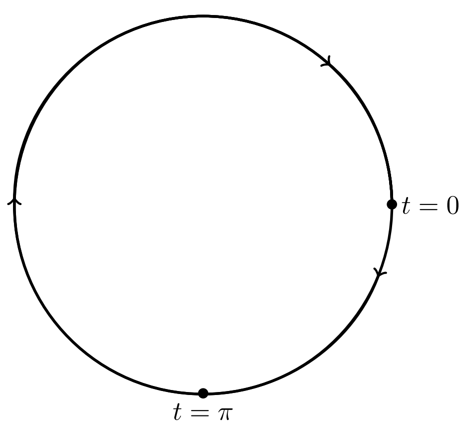
Looking at the curve traced out over any interval of time longer that 2π will indeed trace out the entire circle. Our "etch-a-sketch" just continues winding around the circle in a counter clockwise direction indefinitely as time goes on.



What happens if we tweak the parametric equation just a bit? With $x = \cos(2t)$ and $y = \sin(2t)$, I still get a picture of a circle. I have just shortened the period of the functions for x and y to π instead of 2π. In other words, this parametrization goes around the circle faster!



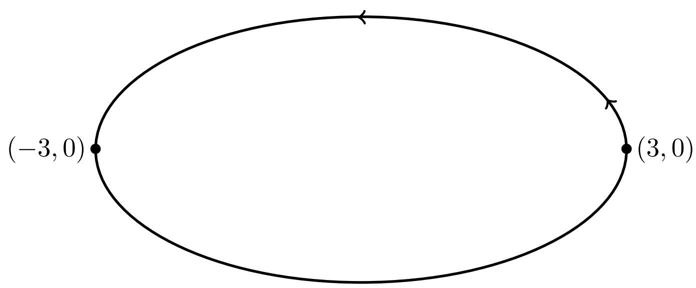
Similarly, a parametrization $x = \cos(-t)$ and $y = \sin(-t)$ still gives the same picture, but now the curve is parametrized in a **clockwise** fashion. In general there are many ways to parametrize the same curve. (We call such a switch in direction a change in orientation).



Suppose I scale the parametrization of the x coordinate by a constant 3.

$$x = 3\cos(t)$$
$$y = \sin(t)$$

Now when $t = 0$, $x = 3$, and for $t = \pi$, $x = -3$. We have a picture that is stretched out into an ellipse!



Note that a parametric equation gives x as a function of time. Each value of t then specifies a **position** along the x axis. Hence we can talk about the first derivative $\frac{dx}{dt}$, and interpret this rate of change as an instantaneous velocity in the x dimension.

Instantaneous velocity in the x dimension is given by

$$\frac{dx}{dt}.$$

Instantaneous velocity in the y dimension is given by

$$\frac{dy}{dt}.$$

Example. Find the instantaneous velocity in the x and y dimensions of the parametric curve

$$\boxed{x = 3t - 4 \qquad y = 2t + e^t.}$$

Solution: It is simple to calculate that

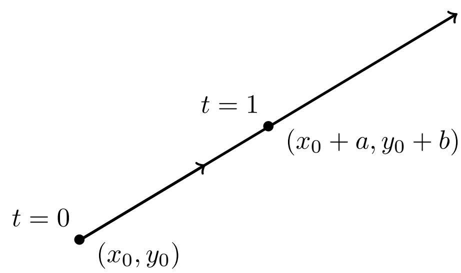
$$\frac{dx}{dt} = 3 \qquad \frac{dy}{dt} = 2 + e^t.$$

We can interpret this to mean that a particle whose motion along the curve is described by the parametrization will be moving to the right (the x direction) at a constant velocity of 3 (units of x per unit of time), while the velocity in the y direction is increasing exponentially.

Parametric Equation of a Straight Line: The motion of a particle along a straight line through a point (x_0, y_0) with slope m can be parametrized by the equations

$$\boxed{x = x_0 + at \qquad y = y_0 + bt \qquad \text{where} \quad m = \frac{b}{a}.}$$

Note that in this situation of a straight line, $\frac{dx}{dt} = a$ and $\frac{dy}{dt} = b$ are constants, with their **ratio** giving the slope of the line. That is, **any** choice of a and b that have the same ratio will draw the same picture, just at different velocities.



Recall that the tangent line to a curve at a point (x_0, y_0) is the straight line through that point with the same slope. The parametrization of straight lines gives us the following important fact.

The **slope** of a parametric curve can be defined by

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)},$$

wherever $\frac{dx}{dt} \neq 0$.

Then a tangent line of a parametric curve has a particularly easy parametric description:

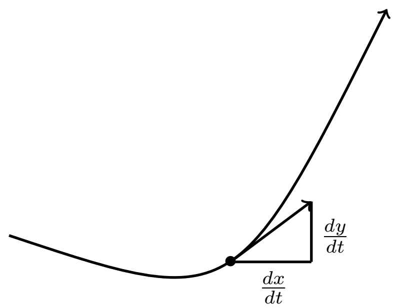
The **tangent line** to a parametric equation $x = x(t)$, $y = y(t)$ at a point $t = a$ is parametrized as

$$x = x(a) + \frac{dx}{dt}\Big|_{t=a} t \qquad y = y(a) + \frac{dy}{dt}\Big|_{t=a} t.$$

We define the **instantaneous speed** of a parametric curve to be

$$S(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Visually, we see that this formula comes from the pythagorean theorem:



Example Recall from above the parametric equation

$$\boxed{x = 3t - 4 \qquad y = 2t + e^t}$$

Find the instantaneous speed at $t = 1$ and parametrize the tangent line to the curve at this point.

Solution: Recall that $\frac{dx}{dt} = 3$ and $\frac{dy}{dt} = 2 + e^t$, so by the formula,

$$S(1) = \sqrt{(3)^2 + (2+e)^2} \approx 5.59.$$

The tangent line is parametrized by

$$x = -1 + 3t \qquad y = 2 + e + (2+e)t$$