Suppose that we needed to know the exact value of the limit

$$\lim_{x\to 0}\frac{x}{e^x-1}, \quad \text{or} \quad \lim_{x\to 0} x\ln|x|, \quad \text{or perhaps} \quad \lim_{x\to 0}((4/5)2^x+(1/5)3^x)^{1/x}.$$

At this point, the only method that we have learned to evaluate limits is to use different algebraic manipulations. Unfortunately, these methods will be of no use to calculate any of the above limits. However, this is a way to compute these limits.

1 L'Hôpital's Rule

Let us first suppose that we are attempting to compute $\lim_{x\to a}\frac{f(x)}{g(x)}$, where $f(a)=g(a)=0$. We cannot simply evaluate the numerator and denominator at $x=a$, since this results in $0/0$, which does not make sense. Remember, local linearization gives the approximations $f(x)\approx f'(a)(x-a)$ and $g(x)\approx g'(a)(x-a)$ for values of x near a. Therefore, if $g'(a)\neq 0$, for values of x near a,

$$\frac{f(x)}{g(x)}\approx\frac{f'(a)(x-a)}{g'(a)(x-a)}=\frac{f'(a)}{g'(a)}.$$

Perhaps then,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}.$$

This is actually the correct intuition. Indeed, since $f(a)=g(a)=0$ and $g'(a)\neq 0$,

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}&=\lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}\\&=\frac{\lim\limits_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)}{\lim\limits_{x\to a}\left(\frac{g(x)-g(a)}{x-a}\right)}=\frac{f'(a)}{g'(a)}.\end{aligned}$$

We have just proved the following theorem:

Theorem (*L'Hôpital's Rule*). *If f and g are differentiable functions, $f(a)=g(a)=0$, and $g'(a)\neq 0$, then*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}.$$

Example 1. Use l'Hôpital's rule to evaluate $\lim\limits_{x\to 0}\dfrac{x}{e^x-1}=1$.

Solution. Let $f(x)=x$ and $g(x)=e^x-1$. Since $f(0)=g(0)=0$ and $g'(0)=e^x|_{x=0}=1$, we have

$$\lim_{x\to 0}\frac{x}{e^x-1}=\lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{f'(0)}{g'(0)}=\frac{1}{e^0}=1.$$

□

If $f'(a)=g'(a)=0$ as well, we may iterate on l'Hôpital's rule. That is, the following result is also true:

Theorem (*More General Form of l'Hôpital's Rule*)**.** *If f and g are differentiable and $f(a) = g(a) = 0$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists.

Example 2. Calculate $\lim_{t\to 0} \frac{e^t - 1 - t}{t^2}$.

Solution. Let $f(t) = e^t - 1 - t$ and let $g(t) = t^2$. Then $f(0) = e^0 - 1 = 0$ and $g(0) = 0$. Furthermore, since $f'(t) = e^t - 1$ and $g'(t) = 2t$, $f'(0) = e^0 - 1 = 0$ and $g'(0) = 0$. Using the above version of l'Hôpital's rule, we have

$$\lim_{t\to 0} \frac{e^t - 1 - t}{t^2} = \lim_{t\to 0} \frac{f(t)}{g(t)} = \lim_{t\to 0} \frac{f'(t)}{g'(t)} = \lim_{t\to 0} \frac{f''(t)}{g''(t)} = \lim_{t\to 0} \frac{e^t}{2} = \frac{1}{2}.$$

□

We can also use use l'Hôpital's rule in cases involving infinity:

Theorem (*Most General Form of l'Hôpital's Rule*)**.** *Assume that f and g are differentiable. For any real number a or $\pm\infty$, when*

$$\lim_{x\to a} f(x) = \pm\infty \quad \textit{and} \quad \lim_{x\to a} g(x) = \pm\infty,$$

or

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0,$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists.

Example 3. Calculate $\lim_{x\to \infty} \frac{x + e^{-2x}}{5x}$.

Solution. Let $f(x) = x + e^{-2x}$ and $g(x) = 5x$. Then $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, so the above theorem applies. We have $f'(x) = 1 - 2e^{-2x}$ and $g'(x) = 5$. So

$$\lim_{x\to\infty} \frac{x + e^{2x}}{5x} = \lim_{x\to\infty} \frac{1 - 2e^{-2x}}{5} = \frac{1}{5}.$$

□

2 Dominance, Powers, Exponentials, and Logarithms

In Chapter 1, we observed that some functions grow much faster than others as $x \to \infty$. We say that g *dominates* f as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. L'Hôpital's rule can be very useful in determining if some function dominates another.

Example 4. Check that $\sqrt{x}$ dominates $\ln x$ as $x \to \infty$.

Solution. Observe that $\lim_{x\to\infty} \sqrt{x} = \lim_{x\to\infty} \ln x = \infty$. Therefore, l'Hôpital's rule implies that

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

Therefore, $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0$, which means that $\sqrt{x}$ dominates $\ln x$ as $x \to \infty$. □

Example 5. Verify that any exponential function of the form e^{kx} with $k > 0$ dominates any power function of the form x^n with $n > 0$ as $x \to \infty$.

Solution. We apply l'Hôpital's rule repeatedly on x^n/e^{kx}:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^n}{e^{kx}} &= \lim_{x\to\infty} \frac{nx^{n-1}}{ke^{kx}} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{k^2e^{kx}} \\
&\vdots \\
&= \lim_{x\to\infty} \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\cdot x}{k^n e^{kx}} \\
&= \lim_{x\to\infty} \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{k^{n+1} e^{kx}} = 0.
\end{aligned}$$

Therefore, e^{kx} dominates x^n as $x \to \infty$ for any real numbers $k, n > 0$. In particular, this shows that exponential functions of this form dominate polynomials as $x \to \infty$. □

3 Recognizing the Form of a Limit

To determine how we should use l'Hôpital's rule, it is convenient to gives names to certain indeterminate forms:

- If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then we say that $\lim_{x\to a} \frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$.
- If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then we say that $\lim_{x\to a} \frac{f(x)}{g(x)}$ has the form $\frac{\infty}{\infty}$.
- If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$, then we say that $\lim_{x\to a} f(x)g(x)$ has the form $\infty \cdot 0$.
- If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$, then we say that $\lim_{x\to a} f(x)^{g(x)}$ has the form ∞^0.
- If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$, then we say that $\lim_{x\to a} f(x)^{g(x)}$ has the form 1^∞.
- If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then we say that $\lim_{x\to a} f(x)^{g(x)}$ has the form 0^0.
- If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then we say that $\lim_{x\to a} \big(f(x) - g(x)\big)$ has the form $\infty - \infty$.

We have already look at indeterminate forms of type $\frac{0}{0}$ and type $\frac{\infty}{\infty}$. Let look at examples of the rest.

Example 6. Rewrite $\lim_{x\to 0^+} x \ln x$, so that you can use l'Hôptial's rule.

Solution. Notice, this limit has the indeterminate form $0 \cdot \infty$. We can rewrite the function so that the limit has the form $\frac{\infty}{\infty}$:

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}},$$

which can be evaluated using l'Hôpital's rule. □

Example 7. Rewrite $\lim_{x\to\infty} x^{1/x}$, so that you can use l'Hôptial's rule.

Solution. This limit has the indeterminate form ∞^0. Since $\ln x$ is a continuous function, we have

$$\ln\left(\lim_{x\to\infty} x^{1/x}\right) = \lim_{x\to\infty} \ln\left(x^{1/x}\right) = \lim_{x\to\infty} \frac{1}{x} \ln x,$$

which can be evaluated using l'Hôpital's rule. To finish the computation, just exponentiate the above limit. In this example, $\lim_{x\to\infty} \ln x/x = 0$, so $\lim_{x\to\infty} x^{1/x} = e^0 = 1$. □

Remark This method can also usually be applied when the limit has the indeterminate form 1^∞ or 0^0, as will be shown in the next two examples.

Example 8. Rewrite $\lim_{x\to 0}((4/5)2^x + (1/5)3^x)^{1/x}$, so that you can use l'Hôpital's rule.

Solution. This limit has the indeterminate form 1^∞. Using a similar technique as in Example 7,

$$\lim_{x\to 0} \ln\left[((4/5)2^x + (1/5)3^x)^{1/x}\right] = \lim_{x\to 0} \frac{\ln((4/5)2^x + (1/5)3^x)}{x},$$

which can be evaluated using l'Hôpital's rule. Exponentiate this limit to finish the computation. □

Example 9. Rewrite $\lim_{x\to 0^+} x^x$, so that you can use l'Hôptial's rule.

Solution. This limit has the indeterminate form 0^0. Using a similar technique as in Example 7,

$$\lim_{x\to 0^+} \ln x^x = \lim_{x\to 0^+} x\ln x,$$

which was considered in Example 6. □

Example 10. Rewrite $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$, so that you can use l'Hôptial's rule.

Solution. Notice, this limit has the indeterminate form $\infty - \infty$. Often times, these limits can be computed by algebraic manipulation. Specifically, in this case, we need only add the fractions:

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{\sin x - x}{x\sin x},$$

which can be evaluated by using l'Hôpital's rule twice. □