Math 1300 Notes:

Section 4.6 - Related Rates

IDEA: In this section, variables are implicitly functions of time.
WANT: To find the derivative with respect to time of one of our variables.
TO SOLVE: Use the relationship between variables, implicit differentiation and the chain rule, and the known derivative of our other variable with respect to time.

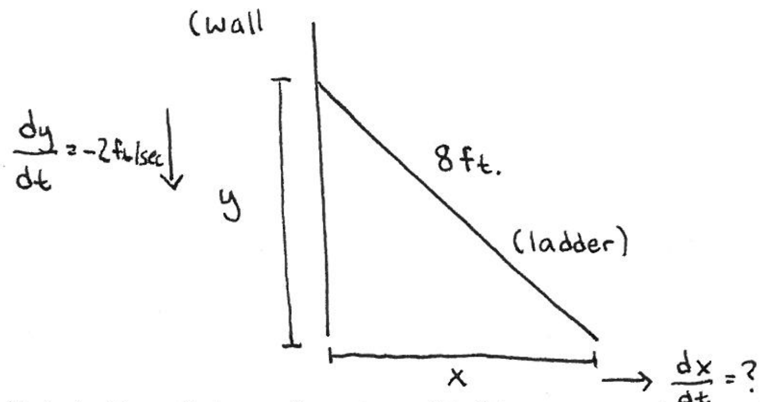
EXAMPLE 1 (with Steps for Solving Related Rates Problems):

An 8 foot long ladder is leaning against a wall. The top of the ladder is sliding down the wall at the rate of 2 feet per second. How fast is the bottom of the ladder moving along the ground at the point in time when the bottom of the ladder is 4 feet from the wall?

Steps:

1. Make a Diagram:

Name and label the variables (quantities that are changing).
Distinguish between "true" constants" and "variable value".



Name/Label: y = distance from top of ladder to ground, x = distance from bottom of ladder to the wall
"True constants": ladder is 8 ft (not changing)
"Variable values": distance from bottom of ladder to wall at time t is 4 ft.

*Do not substitute "variable values" until later [Step 5 - Substitute]

2. Rates:

Identify needed rate of change
Identify known rates of change
*Note: + means increasing, − means decreasing

Needed rate of change: $\frac{dx}{dt}$ when $x = 4$
Known rates of change: $\frac{dy}{dt} = -2\frac{ft}{sec}$
*Because the ladder is sliding down the wall, the y-value is decreasing so we have a negative rate of change

3. Equation Relating Variables:

Find an equation relating the two variables from above.
Use geometry, area or volume formulas, trig formulas, Pythagorean Theorem, physics.
*If the equation you get has more than these two variables, use extra info from the problem to eliminate the other variables.

By the Pythagorean Theorem, $x^2 + y^2 = (8 \text{ ft})^2$.
*It's okay to plug in 8 feet because it's a "true constant". DO NOT substitute $x = 4$ yet, as that's a "variable value"

4. Differentiate:

No need to solve for y before taking the derivative.
Instead, differentiate implicitly (using the chain rule), always with respect to time.
Be sure to use product rule, quotient rule, etc. when necessary!

$$\begin{aligned} \frac{d}{dt}(x^2 + y^2) &= \frac{d}{dt}(8^2) \\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \end{aligned}$$

5. Substitute:

Now is the time to use your "variable values" and known rates of change.
IMPORTANT: Never substitute until after you differentiate!
You might need to re-use the equation relating your variables again as part of substitution.

At the time t we want to find $\frac{dx}{dt}$, the bottom of ladder is 4 ft from wall, so $x = 4$ ft:
$2(4\text{ft})\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

Use Pythagorean Theorem (again) to get that when the bottom of the ladder is 4 ft from the wall, the top of the ladder is $4\sqrt{3}$ ft from ground, so $y = 4\sqrt{3}$ ft:
$2(4\text{ft})\frac{dx}{dt} + 2(4\sqrt{3}\text{ft})\frac{dy}{dt} = 0$

Known rate of change is $\frac{dy}{dt} = -2\frac{\text{ft}}{\text{sec}}$:
$2(4\text{ft})\frac{dx}{dt} + 2(4\sqrt{3}\text{ft})(-2\frac{\text{ft}}{\text{sec}}) = 0$

6. Solve:

Solve for needed rate of change. Check units!

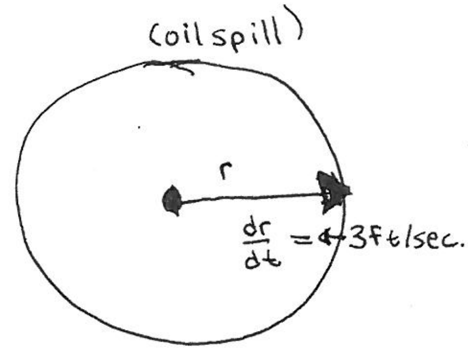
We have: $8\text{ft}\frac{dx}{dt} + -16\sqrt{3}\frac{\text{ft}^2}{\text{sec}} = 0$
So:

$$\begin{aligned} \frac{dx}{dt} &= 16\sqrt{3}\frac{\text{ft}^2}{\text{sec}} \cdot \frac{1}{8\text{ft}} \\ \frac{dx}{dt} &= 2\sqrt{3}\frac{\text{ft}}{\text{sec}} \end{aligned}$$

So the bottom of the ladder is moving along the ground at a rate of $2\sqrt{3}\frac{\text{ft}}{\text{sec}}$ at the time when the bottom of the ladder is 4 feet from the wall.

EXAMPLE 2:

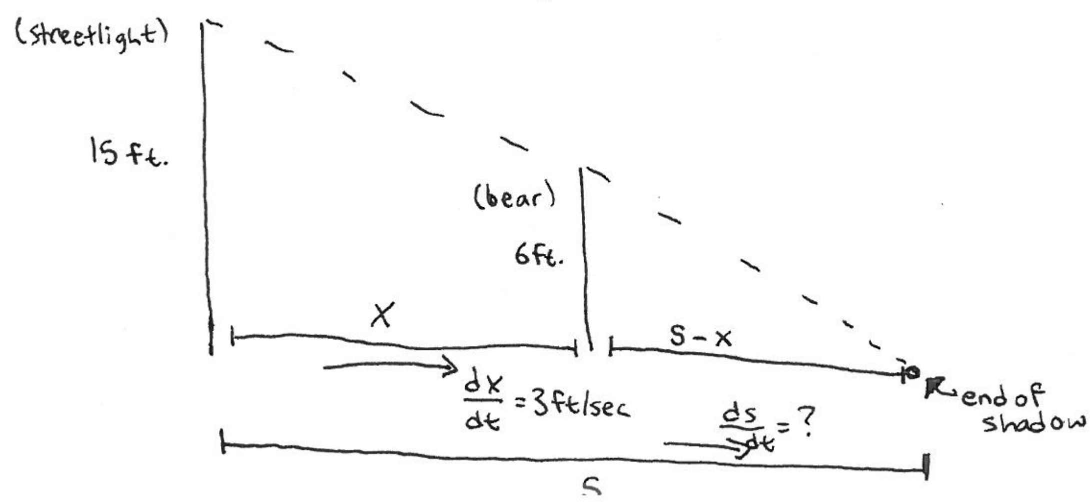
Oil spills out of a tanker in a circular shape whose radius is increasing at a rate of $3\frac{\text{ft}}{\text{sec}}$. How fast is the area of the spill increasing when the radius is 60 ft.?



1. Our variables are the radius r and area A. "Variable values": the radius at time t is 60 ft.
2. Rates:
   Known rate of change: $\frac{dr}{dt} = 3\frac{\text{ft}}{\text{sec}}$.
   Needed rate of change: $\frac{dA}{dt}$
3. Equation relating r, A: $A = \pi r^2$.
4. Differentiate: $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$
5. Substitute: $\frac{dA}{dt} = 2\pi(60\text{ft})(3\frac{\text{ft}}{\text{sec}})$
6. Solve: $\frac{dA}{dt} = 360\pi \frac{\text{ft}^2}{sec}$

EXAMPLE 3:

A streetlight is mounted at the top of a 15 ft pole. A 6 ft. tall bear walks away from the pole at a speed of 3 $\frac{\text{ft}}{\text{sec}}$. How fast is the end of its shadow moving?



1. Our variables are the distance of the bear from the pole - call that "x" - and the distance from the end of its shadow from the pole - call that "s".
2. Rates:
   Known rate of change: $\frac{dx}{dt} = 3\frac{\text{ft}}{\text{sec}}$.
   Needed rate of change: $\frac{ds}{dt}$
3. Equation relating x, s:
   By similar triangles, we get $\frac{s-x}{6} = \frac{s}{15}$.
   So $15s - 15x = 6s$, and we get $9s = 15x$.
4. Differentiate:

   $\frac{d}{dt}[9s] = \frac{d}{dt}[15x]$
   $9\frac{ds}{dt} = 15\frac{dx}{dt}$

5. Substitute: $9\frac{ds}{dt} = 15(3\frac{\text{ft}}{\text{sec}})$
6. Solve: $\frac{ds}{dt} = 5\frac{\text{ft}}{\text{sec}}$.