MATH 1300
SECTION 4.4: OPTIMIZATION, MODELING, GEOMETRY

(We will assume all functions are continuous on their domain.)

RECALL

§4.1 how to find local extrema

- Find the critical points of the function by solving
  $f'(x) = 0$ or DNE.
- FIRST DERIVATIVE TEST
  Find the sign of $f'(x)$ on the intervals given by the critical points:

$$\xleftarrow{\quad + \quad} \text{(max)} \xrightarrow{\quad - \quad} \text{(min)} \xrightarrow{\quad + \quad} \text{(neither)} \xrightarrow{\quad + \quad}$$

or

- SECOND DERIVATIVE TEST
  Check concavity at the critical points p

$$\begin{aligned} f''(p) > 0 &\quad \text{local min} \\ f''(p) < 0 &\quad \text{local max} \\ f''(p) = 0 &\quad \text{inconclusive!} \end{aligned}$$

§4.2 Optimization: finding GLOBAL extrema

- When there is only ONE CRITICAL POINT
  Test for a local extreme at the critical point.
- On a CLOSED INTERVAL [a,b]
  Compare $f(a)$, $f(b)$ and local extrema.
- On an OPEN INTERVAL and all real numbers $(-\infty, \infty)$
  Compare local extrema and the end behavior of the function

Date: 10/28-29.

$\boxed{\text{Motivating Example.}}$

Suppose $G(s)$ below gives the fuel consumption (in mpg) of a car traveling at a speed s (in mph).

$$G(s) = -\frac{1}{20}(s-30)^2 + 45.$$

(1) What is the global maximum of G over the interval [0, 25]?

Since the derivative exists everywhere, the only critical points are solutions to

$$\begin{aligned} G'(s) &= 0 \\ -\frac{1}{10}(s-30) &= 0 \\ s &= 30 \end{aligned}$$

This is then the **only** critical point of G, and it is not in our interval [0,25].

So G must have a maximum at an endpoint. Comparing

$$G(0) = -\frac{1}{20}(0-30)^2 + 45 = 0 \qquad \text{and} \qquad G(25) = -\frac{1}{20}(25-30)^2 + 45 = 43.25,$$

we see that on the interval $[0, 25]$, G has a $\boxed{\text{maximum of 43.25 mpg at a speed of 25 mph.}}$

What did this calculation tell us? We have used the function describing fuel consumption to determine that in a 25 mph zone, it is best to drive the speed limit rather than slower. But is 25 mph the best we can do in general?

(2) What is the global minimum of G on $(-\infty, \infty)$?

$(-\infty, \infty)$ is an open interval, and $s = 30$ is the only critical point, so we apply the first derivative test to see it is local maximum:

$$(G') \qquad \longleftarrow\!\xrightarrow{+}\; 30 \;\xrightarrow{-}\!\longrightarrow$$

Thus $s = 30$ mph gives a local (and global) maximum for G of $G(30) = 45$ mpg.

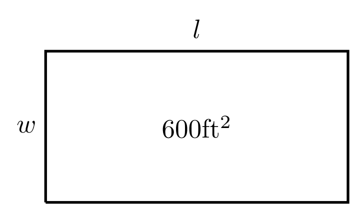
We have answered an important question: Given a function describing fuel consumption, what is the most fuel efficient speed at which I should drive my car?

This is a simple example of the use of optimization and modeling. In general, there may be a real world situation that we would like to optimize (the optimum route to take to drive to work, the optimum dimensions for a can of soda, the optimum price at which to sell airline tickets during the holidays, etc.) The goal is to come up with a function that describes the situation we want to optimize, and then use the techniques of §4.2.

Often times, the quantity we wish to optimize is subject to some **constraints**. For instance, above we found that the best speed at which to drive the car was 30 mph, but subject to the *constraint* that my speed must be between 0 and 25 mph, 25 mph was best.

Example. An elementary school needs to fence in an area of land for a rectangular playground. The playground must be 600 square feet, and one side bordering a road must be built with a taller fencing. If the cost of the regular fencing is $2 per foot, and the taller fencing costs $4 per foot, what dimensions should the school choose for its playground?

(1) Draw a picture:



(2) What are we trying to **optimize**?

We should try to minimize the cost of the fencing! So we need a function that describes the **cost in terms of the dimensions**. Since we are fencing in a rectangular portion, we can use ℓ for the length and w for the width. We may as well assume that the more expensive side will occur in the width, so we come up with an equation for cost:

$$C = 2(\$2)\ell + \$2w + \$4w,$$

or written in a simpler way,

$$\boxed{C = 4\ell + 6w \quad \text{dollars.}}$$

(3) Now, what **constraints** do we have?

We need the rectangle to have an **area equal to** 600 **square feet**, and we also know that physically ℓ and w must take on values that are positive. We can express this by writing

$$\boxed{\ell w = 600 \quad \text{and} \quad \ell, w > 0.}$$

(4) Use the constraints to rewrite C as an explicit function of one variable w.

Since $\ell w = 600$, we know that

$$\ell = \frac{600}{w},$$

so we substitute and get

$$\boxed{C = \frac{4(600)}{w} + 6w.}$$

(5) Optimize your function!

We need to find the **global minimum** of C on the open interval $(0, \infty)$. The critical point will occur when

$$\begin{aligned} C' = \frac{-4(600)}{w^2} + 6 &= 0 \\ 6 &= \frac{4(600)}{w^2} \\ 6w^2 &= 4(600) \\ w^2 &= \frac{4(600)}{6} \\ w &= \pm 20. \end{aligned}$$

Since our interval is $(0, \infty)$, we get only **one critical point**, $w = 20$.

Using the **second derivative test**, we see that at $w = 20$,

$$C'' = \frac{8(600)}{20^3} > 0$$

so C is concave up here and we have a local (so also global) minimum at $\boxed{\text{w=20}}$ on the interval $(0, \infty)$. Finally, don't forget that

$$\ell = \frac{600}{w},$$

so $\boxed{\ell = 30}$ feet. Hence we have found that the **cost minimizing dimensions will be** $\boxed{20 \times 30 \text{ feet}}$, with a 20 foot side along the road.

If we want to know what was the cost that these dimension got us, we can just plug into the original cost function to find that the **minimum cost** is

$$\boxed{4(30) + 6(20) = \$240.}$$

It is important to note that the hardest part about optimization problems is the need to write down the functions yourself just from the information given in the problem. A suggested method on how to do this might look something like this:

Steps for Optimizing

- Draw a picture of the situation (if possible)
- Identify **what you need to optimize** (cost, area, volume, time, etc.)
- Write a formula for the **variable to be optimized** as well as formulae describing constraints.
- Substitute the constraints into the formula for the **variable to be optimized** in order to get the **variable to be optimized** as a function of one other variable.
- Identify the domain over which you are optimizing.
- Find the extrema of your function over the domain.

Useful Geometric Identities

(b = base; h = height, ℓ = length, w = width, r = radius, s = hypotenuse)

- **Triangle**

$$\text{Area} = \frac{1}{2} b \cdot h$$

- **Rectangle**

$$\text{Perimeter} = 2\ell + 2w \qquad \text{Area} = \ell \cdot w$$

- **Circle**

$$\text{Circumference} = 2\pi r \qquad \text{Area} = \pi r^2$$

- **Sphere**

$$\text{Surface Area} = 4\pi r^2 \qquad \text{Volume} = \frac{4}{3}\pi r^3$$

- **Rectangular Box**

$$\text{Surface Area} = 2\ell \cdot w + 2\ell \cdot h + 2w \cdot h \qquad \text{Volume} = \ell \cdot w \cdot h$$

- **Cylinder**

$$\text{Surface Area} = 2\pi r^2 + 2\pi r \cdot h \qquad \text{Volume} = \pi r^2 h$$

- **Cone**

$$\text{Surface Area} = \pi r^2 + \pi r \cdot s \qquad \text{Volume} = \frac{1}{3}\pi r^2 h$$