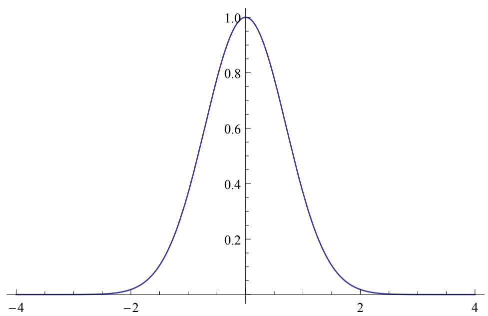
Section 4.3 - Families of Functions

- Often times it is useful to consider a group of functions which are so similar that they only differ by constants. For example, $f(x) = ae^{kx}$ specifies a different function for each choice of a and k, but by leaving a and k as unknowns, we can find the roots, y-intercepts, critical points, and inflection points of all exponential functions at once rather than working with each exponential function individually. Below we examine several common families of functions. The important thing to take away is the process we use to examine the functions rather than the results of the examination.

- **The Bell-Shaped Curve**: A common family of functions in statistics is the family of *normal density functions* $f(x) = e^{-(x-a)^2/b}$ for $b > 0$. Graphically, for the case where $a = 0$ and $b = 1$:



From the graph, we can see that $f(x)$ has one maximum and two inflection points. It is possible, though tedious, to verify that this holds for all choices of a and b. We wish to find the x-values of the maximum and inflection points of $f(x)$. To this end, we find the first and second derivatives of $f(x)$:

$$f'(x) = \frac{-2(x-a)}{b}e^{-(x-a)^2/b}, f''(x) = \frac{4(x-a)^2}{b^2}e^{-2(x-a)^2/b} + \frac{-2}{b}e^{-(x-a)^2/b}.$$

Setting both of these equal to zero,

$$\frac{-2(x-a)}{b}e^{-(x-a)^2/b} = 0 \text{ and } \frac{4(x-a)^2}{b^2}e^{-2(x-a)^2/b} + \frac{-2}{b}e^{-(x-a)^2/b} = 0.$$

In the first equation, we note that $e^{-(x-a)^2/b}$ is never zero, so we must have $-2(x-a)/b = 0$, which we solve to get $x = a$. In the second equation, we factor out an $e^{-(x-a)^2/b}$ to get

$$(\frac{4(x-a)^2}{b^2} + \frac{-2}{b})e^{-(x-a)^2/b} = 0.$$

Again, $e^{-(x-a)^2/b}$ is never zero, so we have $\frac{4(x-a)^2}{b^2} + \frac{-2}{b} = 0$, which we solve to get $x = a \pm \sqrt{b/2}$, so that the maximum of $f(x)$ is at $x = a$ and the inflection points are at $a \pm \sqrt{b/2}$. From this we can see that a specifies the center of the bell and that b determines how wide the bell is (larger values of b give wider bells).

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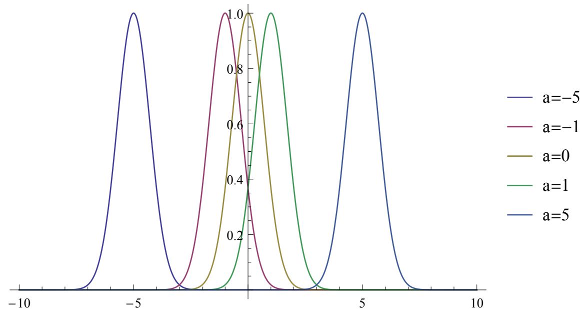

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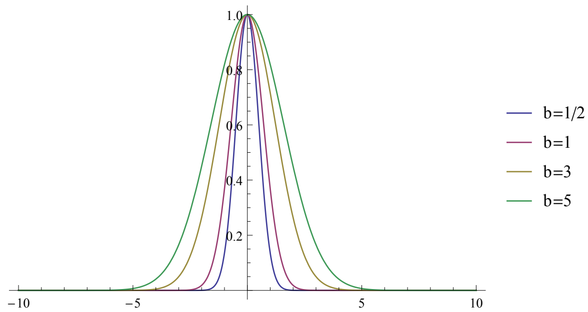
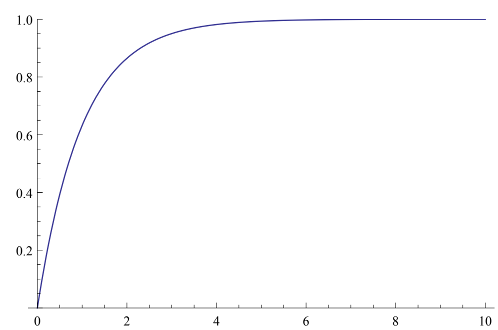


- **The Exponential Model with a Limit**: A common family of functions in modeling free fall in a fluid is the family of *limited exponential functions*: $f(x) = a(1 - e^{-bx})$, where $a, b > 0$.



Fixing a and graphing $f(x)$ for various values of b, we see that, for a given value of a, b controls the initial steepness of the graph. We verify this by evaluating f' at zero:

$$\begin{aligned} f'(0) &= abe^{-b(0)} \\ &= ab \cdot 1 \\ &= ab. \end{aligned}$$

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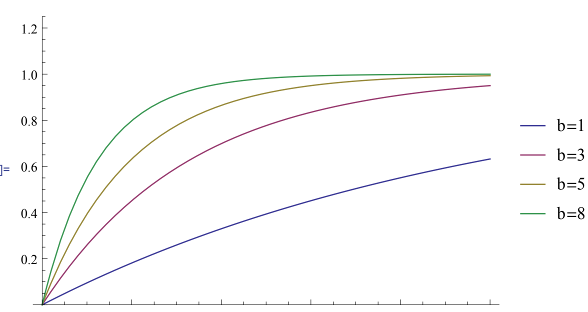
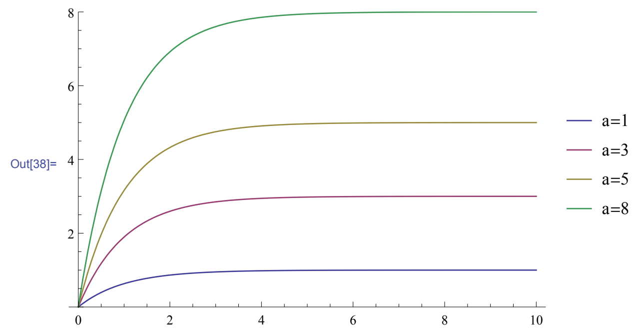


Taking the limit of $f(x)$ as $x \to \infty$,

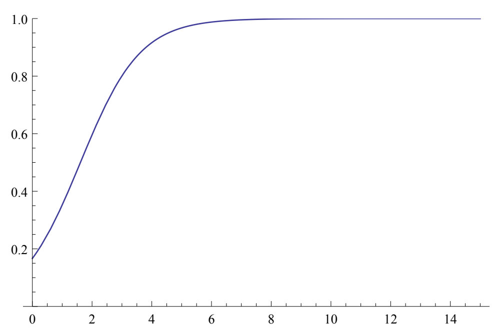
$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} a - ae^{-bx} \\ &= a - a \lim_{x\to\infty} e^{-bx} \\ &= a - a \cdot 0 \\ &= a,\end{aligned}$$

so we see that $y = a$ is the horizontal asymptote for $f(x)$.



In terms of free fall, ab is the initial velocity and a is the object's terminal velocity in the fluid.

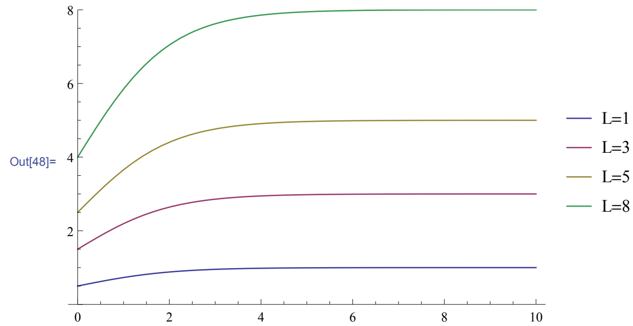
- **The Logistic Model**: A common family of functions used to model population growth when there is an environmentally imposed maximum population is the family of *logistic functions*: $f(t) = L/(1 + Ae^{-kt})$ for $L, A, k > 0$.



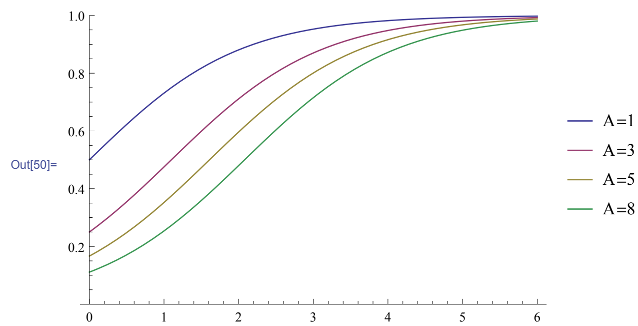
Taking the limit of $f(t)$ as $t \to \infty$,

$$\begin{aligned}\lim_{t\to\infty} f(t) &= \lim_{t\to\infty} \frac{L}{1 + Ae^{-kt}} \\ &= \frac{L}{1 + A \lim_{t\to\infty} e^{-kt}} \\ &= \frac{L}{1 + A \cdot 0} \\ &= L,\end{aligned}$$

and we see that $y = L$ is the horizontal asymptote for $f(t)$.



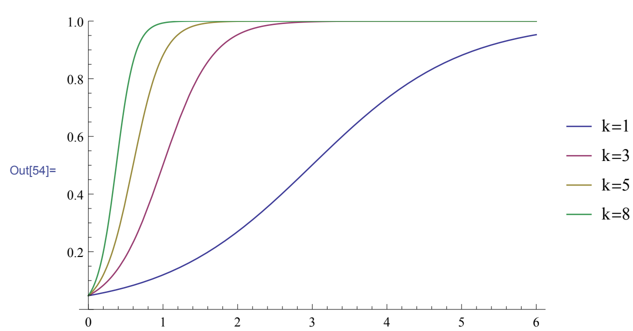
In terms of population modeling, L is the maximum population the environment can hold. Fixing L and k and looking at $f(0)$ for various values of A, we can see that A determines the initial population once the limiting population is known. We verify this by evaluating $f(0) = L/(1+A)$.



Differentiating $f(t)$, we get

$$f'(t) = \frac{LAke^{-kt}}{(1+Ae^{-kt})^2}.$$

So, we see that $f'(0) = LAk/(1+A)^2$, so that given a starting population, k determines the initial population growth rate.



Finally, examining the second derivative,

$$f''(t) = \frac{Ae^{kt}(A-e^{kt})k^2L}{A+e^{kt^3}}$$

we can see that $f(t)$ has an inflection point when $A - e^{kt_0} = 0$, which occurs when $t_0 = \frac{\ln A}{k}$. At this value of t_0,

$$\begin{aligned} f(t_0) &= \frac{L}{1 + Ae^{-k\frac{\ln A}{k}}} \\ &= \frac{L}{1 + Ae^{-\ln A}} \\ &= \frac{L}{1 + A(e^{\ln A})^{-1}} \\ &= \frac{L}{1 + A \cdot A^{-1}} \\ &= \frac{L}{1+1} &= \frac{L}{2}. \end{aligned}$$

So, the rate of population growth starts decreasing at half the limiting population.

- For an arbitrary family of functions, we will generally want to determine the critical points, the inflection points, and any asymptotes of the family in terms of the family's parameters, as we have done in the three examples above.