1. Section 4.1 of HH - Using First and Second Derivatives

   (a) Local Maxima and Minima (extrema). f has a local minimum/maximum at p if $f(p)$ is less than or equal to/greater than or equal to values of f for points near p.

   (b) Visually, it seems reasonable to guess that local extrema occur at points where the tangent line is horizontal (valleys or mountains in the graph). However, we have to be careful because of functions like $f(x) = |x|$. The tangent lines to this function are never horizontal, but it *does* have a relative minimum.

   (c) So, we define *critical points*. A critical point is a point p in the domain of $f(x)$ where either $f'(p) = 0$ or $f'(p)$ does not exist.

   (d) Local extrema occur at critical points. *NOTE: Not all critical points correspond to local extrema.* Take $f(x) = x^3$ at $x = 0$. Certainly $x = 0$ is a critical point, but $f(x)$ has no minimum or maximum there.

   (e) The moral: at a critical point, f may have a maximum, a minimum, or neither. Simply looking at the derivative at one point does not tell us enough information to decide which of the three is the case. To classify critical points as maxima, minima, or neither, we can use the first or second derivative tests.

2. The First Derivative Test

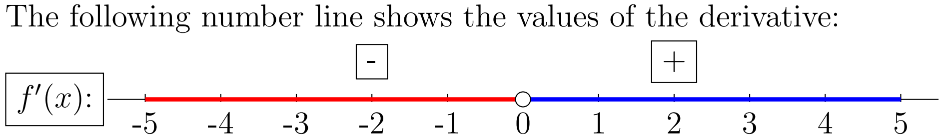
   - Suppose p is a critical point of a continuous function f
   - If f' changes from negative to positive at p, then f has a local minimum at p
   - If f' changes from positive to negative at p, then f has a local maximum at p

3. Examples

   (a) Let $f(x) = |x|$, (a continuous function). Find the critical points, and use the first derivative test to classify them as maxima, minima, or neither.

   We know from our previous study of the absolute value function that $f'(0)$ does not exist. For $x < 0$, $f'(x) = -1$, and for $x > 0$, $f'(x) = 1$. Thus, $x = 0$ is the only critical point.

   The following number line shows the values of the derivative:



   and we can see that f' changes from negative to positive at $x = 0$, so f has a local minimum at 0.

   (b) Let $g(x) = 3x^4 - 8x^3$. Find the critical points, and use the first derivative test to classify them as maxima, minima, or neither.

   First, take the derivative: $g'(x) = 12x^3 - 24x^2$.

   Then, we should ask two questions:

i. Where is $g'(x) = 0$?

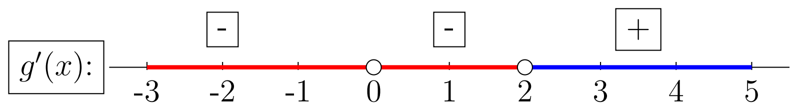
ii. Where does $g'(x)$ not exist?

We can solve $g'(x) = 0$, and since $g'(x)$ is a polynomial, it is defined on all real numbers.

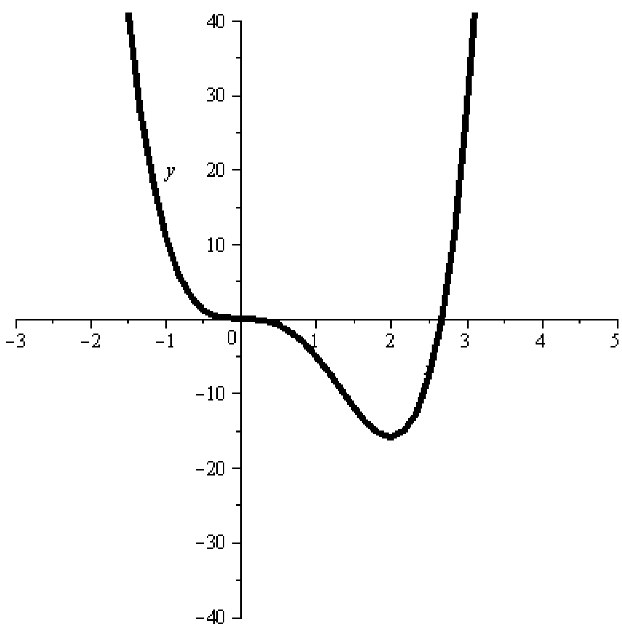
Now, solve $12x^3 - 24x^2 = 0$.

Factoring, we get $12x^2(x-2) = 0$. This gives us two critical points: $x = 0$ and $x = 2$. We should use the first derivative test to check the values of the derivative between the critical points. For test points, we can check $g'(-1) = -36$, $g'(1) = -12$, and $g'(3) = 108$.

Since the sign of $g'(x)$ does not change at $x = 0$, g has neither a maximum nor a minimum at $x = 0$, and at $x = 2$, $g'(x)$ changes from negative to positive, so g has a local minimum.



We can verify this by looking at the graph of g itself:



4. The Second Derivative Test

   The second derivative gives us an alternate method of telling whether a critical point is a local minimum, local maximum, or neither.

   - Suppose $f'(p) = 0$
   - If $f''(p) > 0$, then f has a local minimum at p
   - If $f''(p) < 0$, then f has a local maximum at p
   - IMPORTANT: If $f''(p) = 0$, then the test tells us nothing

5. Examples

   (a) Let $f(x) = x - e^x + 1$. Find the critical points and classify them as maxima, minima, or neither using the second derivative test.

   First, we take the derivative: $f'(x) = 1 - e^x$. This is defined on all real numbers, so the only critical points we will get will be from solving $f'(x) = 0$.

   Solving, we get $x = 0$. Then we can check the concavity at $x = 0$ by taking the second derivative:

   $f''(x) = -e^x$, and therefore $f''(0) = -1$. By the second derivative test, the fact that f is concave down means we have a local maximum at $x = 0$.

   (b) WARNING EXAMPLE: Let $g(x) = x^4$. Find the critical points and classify them as maxima, minima, or neither.

   The derivative is $g'(x) = 4x^3$, and the only critical point is at $x = 0$.

   Can we use the second derivative test? Let's try. $g''(x) = 12x^2$, and at $x = 0$, I get $g''(0) = 0$. This DOES NOT mean that g does not have a max or a min at $x = 0$. This means that the second derivative test is inconclusive. We should use the first derivative test instead.

   For $x > 0$, $g'(x) > 0$, and for $x < 0$, $g'(x) < 0$, so the derivative changes from negative to positive at $x = 0$, which means g has a local minimum at $x = 0$.

6. Inflection Points

   (a) By analogy, if local extrema are where the first derivative changes sign (from positive to negative, or negative to positive), then *inflection points* are where the second derivative changes sign (i.e., where a function changes concavity).

   (b) We can look for inflection points in a similar way as extrema. Inflection points occur where either $f''(p) = 0$ or $f''(p)$ does not exist.

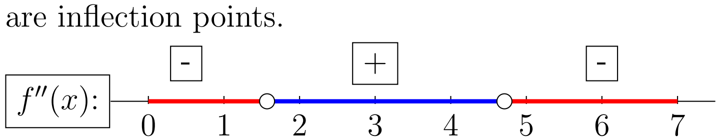
7. Examples

   (a) Let $f(x) = \cos x$ on $[0, 2\pi]$. Find the inflection points.

   Taking derivatives, we get $f'(x) = -\sin x$, and $f''(x) = -\cos x$. This is defined everywhere on $[0, 2\pi]$, so the only inflection points (if any) will occur where $f''(x) = 0$.

   Solving $-\cos x = 0$, we get $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. To see if these are inflection points or not, we should check to see if the second derivative changes sign.

   For test points, we can check values in between where $f''(x) = 0$. So, $-\cos(0) < 0$, $-\cos(3) > 0$, and $-\cos(6) < 0$, so we can conclude that both $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ are inflection points.



We can verify this by looking at the original graph. It changes concavity at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$: