Section 3.9 - Linear Approximation and the Derivative

Idea: When we zoom in on a smooth function and its tangent line, they are very close together. So for a function that is hard to evaluate, we can use the tangent line to approximate values of the derivative.

Example Use a tangent line to approximate $\sqrt[3]{8.1}$ without a calculator.

Let $f(x) = \sqrt[3]{x} = x^{1/3}$. We know $f(8) = \sqrt[3]{8} = 2$. We'll use the tangent line at $x = 8$.

$$f'(x) = \frac{1}{3}x^{-2/3} \quad \text{so} \quad f'(8) = \frac{1}{3}(8)^{-2/3} = \frac{1}{3} \cdot \frac{1}{4}$$

Our tangent line passes through $(8, 2)$ and has slope $1/12$:

$$y = \frac{1}{12}(x-8) + 2.$$

To approximate $f(8.1)$ we'll find the value of the tangent line at $x = 8.1$.

$$\begin{aligned} y &= \frac{1}{12}(8.1-8)+2 \\ &= \left(\frac{1}{12}\right)(.1)+2 \\ &= \frac{1}{120}+2 \\ &= \frac{241}{120} \end{aligned}$$

So $\sqrt[3]{8.1} \approx \frac{241}{120}$.

How accurate was this approximation? We'll use a calculator now: $\sqrt[3]{8.1} \approx 2.00829885$. Taking the difference, $2.00829885 - \frac{241}{120} \approx -3.4483 \times 10^{-5}$, a pretty good approximation!

General Formulas for Linear Approximation

The tangent line to $f(x)$ at $x = a$ passes through the point $(a, f(a))$ and has slope $f'(a)$ so its equation is

$$y = f'(a)(x-a) + f(a)$$

The tangent line is the best linear approximation to $f(x)$ near $x = a$ so

$$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$$

this is called the <u>local linear approximation</u> of $f(x)$ near $x = a$. The <u>error function</u> corresponding to this approximation is

$$\boxed{E(x) = f(x) - f(a) - f'(a)(x-a)}$$

Note: If $f(x)$ is concave down the tangent line an an over-approximation, $E(x) < 0$. If $f(x)$ is concave up the tangent line is an under-approximation, $E(x) > 0$.

Example Find the local linearization of $f(x) = \ln(x)$ near $x = 1$, and find the error function.

Using our formula, $f(x) \approx f(a) + f'(a)(x - a)$. So we have,

$$\begin{aligned} a &= 1 \\ f(a) &= f(1) = \ln(1) \\ f'(x) &= \frac{1}{x} \to f'(a) = \frac{1}{1} = 1. \end{aligned}$$

Therefore, the local linearization is $f(x) \approx 0 + 1(x - 1) = x - 1$. To find the error function:

$$\begin{aligned} E(x) &= f(x) - f(a) - f'(a)(x - a) \\ &= \ln(x) - 0 - 1(x - 1) \\ &= \ln(x) - x + 1 \end{aligned}$$

Theorem

$$\lim_{x \to a} \frac{E(x)}{x - a} = 0.$$

Proof.

$$\begin{aligned} \lim_{x \to a} \frac{E(x)}{x - a} &= \lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} \\ &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} - \lim_{x \to a} \frac{f'(a)(x - a)}{x - a} \\ &= f'(a) - f'(a) = 0 \end{aligned}$$

□

In the next example we will find another local linear approximation. Then we'll investigate analytically how accurate the line is in estimating the curve. We'll need the second derivative to do this.

Example Find the linear approximation to $f(x) = 2x^3 - 4x^2 + 5x - 7$ at $x = 1$. At $x = 1$ we have:

$$\begin{aligned} f(x) &= 2x^3 - 4x^2 + 5x - 7 & f(1) &= -4 \\ f'(x) &= 6x^2 - 8x + 5 & f'(1) &= 3 \\ f''(x) &= 12x - 8 & f''(1) &= 4 \end{aligned}$$

So the linear approximation is $f(x) \approx f(a) + f'(a)(x - a) = -4 + 3(x - 1)$. Finding the error in this approximation:

$$\begin{aligned} E(x) &= f(x) - f(a) - f'(a)(x - a) = -4 + 3(x - 1) \\ &= (2x^3 - 4x^2 + 5x - 7) - (-4) - 3(x - 1) \\ &= 2x^3 - 4x^2 + 5x - 7 + 4 - 3x + 3 \\ &= 2x^3 - 4x^2 + 2x \\ &= 2x(x^2 - 2x + 1) \\ &= 2x(x - 1)^2 \end{aligned}$$

Note that

$$\lim_{x\to 1} \frac{E(x)}{x-1} = \lim_{x\to 1} \frac{2x(x-1)^2}{x-1} = \lim_{x\to 1} 2x(x-1) = 0$$

$$\lim_{x\to 1} \frac{E(x)}{(x-1)^2} = \lim_{x\to 1} 2x = 2 = \frac{4}{2} = \frac{f''(1)}{2}.$$

This leads us to the following theorem:

Theorem

$$\lim_{x\to a} \frac{E(x)}{(x-a)^2} = \frac{f''(a)}{2} \text{ and therefore } E(x) \approx \frac{f''(a)}{2}(x-a)^2.$$