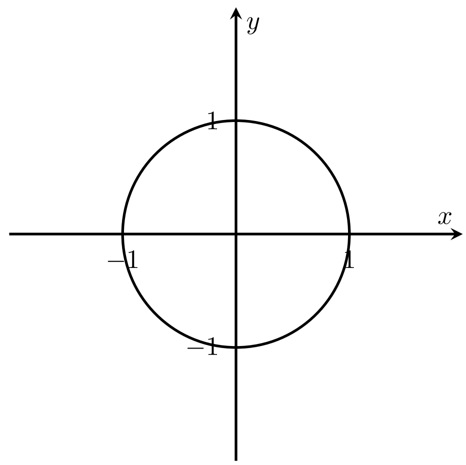
Sec. 3.7: Implicit Functions

Up until now we have mostly represented functions in the form $y = f(x)$ where $f(x)$ is some function defined with a single variable x. In this case y is said to be an *explicit function* of x.

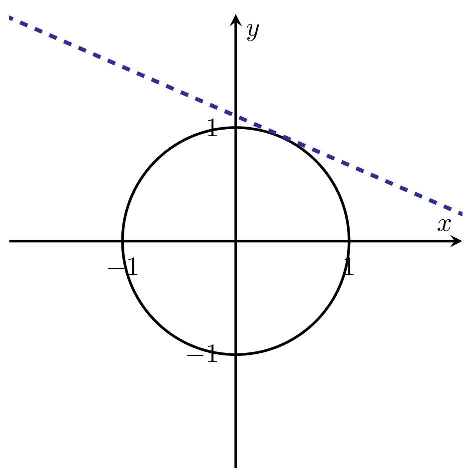
But sometimes we are given an equation where we can't isolate the variable y to be in the above form. Consider the following graph of the unit circle.



This curve is represented by the equation $y^2 + x^2 = 1$. Note that this equation is not a function, you can see this by applying the vertical line test, or by simply noting that at $x = 0$, y is both 1 and -1.

Now we can describe the upper half of this circle by the function $y = \sqrt{1 - x^2}$, and the lower half of this circle by the function $y = -\sqrt{1 - x^2}$. But since the curve $y^2 + x^2 = 1$ isn't a function, there is no way to express it as an explicit function $y = f(x)$. So while the curve $y^2 + x^2 = 1$ isn't itself a function, we say it implies the functions $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$. So an equation of this form is said to give y as an *implicit function* of x.

Now looking at the graph below, you can see that at any point on the curve, there exists line that is tangent to the curve at that point,



So while the curve $y^2 + x^2 = 1$ isn't a function, we still have lines tangent to the curve whose slopes represent the rate of change $\frac{dy}{dx}$ at a point. So we want to find the derivative $\frac{dy}{dx}$ that will give us the slope of the tangent line at a point.
In order to find the derivative of the curve $y^2 + x^2 = 1$, let's first take the derivative of both sides

$$\tfrac{d}{dx}[x^2 + y^2] = \tfrac{d}{dx}[1]$$

Now by the sum rules for derivatives I can break up the left hand side of the equation across the addition. And by the constant rule we know that the right hand side of the equation is equal to 0. So,

$$\tfrac{d}{dx}[x^2] + \tfrac{d}{dx}[y^2] = 0 \qquad \Rightarrow \qquad 2x + \tfrac{d}{dx}[y^2] = 0 \qquad \Rightarrow \qquad \tfrac{d}{dx}[y^2] = -2x$$

Note: Even though y isn't a function of x, it is an implicit function of x. So we still need to apply the chain rule when differentiating y^2

So using the chain rule we have that,

$$\tfrac{d}{dx}[y^2] = 2y\tfrac{dy}{dx}$$

Plugging this into what we already have we see that,

$$2y\tfrac{dy}{dx} = -2x$$

Now remember, we want the function $\frac{dy}{dx}$, so isolating that term we get,

$$\tfrac{dy}{dx} = \tfrac{-2x}{2y} \qquad \Rightarrow \qquad \boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

One thing you might notice is that at $y = 0$ this function is undefined. This makes sense since if you look at the graph you'll notice that at our two points (-1,0), (1,0), where $y = 0$, we will have vertical tangent lines, which have undefined slope.

Note: One question you might ask is what does this derivative $\frac{dy}{dx}$ tell us about our implied functions $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$. The answer is that this formula for the derivative will in fact hold for both of our implied functions. Which we will check below by differentiating the explicit formulas for y.

-Upper half of circle: $\qquad y = \sqrt{1-x^2} \qquad \Rightarrow \qquad y = (1-x^2)^{\frac{1}{2}}$

By by the chain rule, we know that,

$$\tfrac{dy}{dx} = \tfrac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot -2x \qquad \Rightarrow \qquad \tfrac{dy}{dx} = -\tfrac{x}{\sqrt{1-x^2}}$$

Now recall that we defined $y = \sqrt{1-x^2}$,

$$\text{Therefore} \qquad \tfrac{dy}{dx} = -\tfrac{x}{y} \qquad \checkmark$$

-Lower half of circle: $y = -\sqrt{1-x^2} \quad \Rightarrow \quad y = -(1-x^2)^{\frac{1}{2}}$

By by the chain rule, we know that,

$$\tfrac{dy}{dx} = -\tfrac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot -2x \qquad \Rightarrow \qquad \tfrac{dy}{dx} = \tfrac{x}{\sqrt{1-x^2}}$$

Now recall that we defined $y = -\sqrt{1-x^2}$ which implies $\sqrt{1-x^2} = -y$,

$$\text{Therefore} \qquad \tfrac{dy}{dx} = -\tfrac{x}{y} \qquad \checkmark$$

So we can see that given an implicit function y of x and its derivative $\frac{dy}{dx}$, this derivative will hold everywhere on the curve.

Example 1: Let $x^2 + 5x - 4y^2 + 3y = 0$, Find the derivative of this curve $\frac{dy}{dx}$

Solution: Taking the the derivative of both sides we see that,

$$\tfrac{d}{dx}[x^2 + 5x - 4y^2 + 3y] = \tfrac{d}{dx}[0]$$

$$\Rightarrow \qquad \tfrac{d}{dx}[x^2] + \tfrac{d}{dx}[5x] - \tfrac{d}{dx}[4y^2] + \tfrac{d}{dx}[3y] = 0$$

$$\Rightarrow \qquad 2x + 5 - 4\tfrac{d}{dx}[y^2] + 3\tfrac{d}{dx}[y] = 0$$

Now the important thing to remember when taking the derivative with respect to x of any function y (in this case y^2 and y) is that we have to treat y as a function of x. So we have to apply the chain rule,

$$2x + 5 - 4[2y\tfrac{dy}{dx}] + 3[\tfrac{dy}{dx}] = 0$$

Solving for $\frac{dy}{dx}$ we see that,

$$-4[2y\tfrac{dy}{dx}] + 3[\tfrac{dy}{dx}] = -2x - 5 \qquad \Rightarrow \qquad \tfrac{dy}{dx}[-8y+3] = -2x - 5 \qquad \Rightarrow \qquad \boxed{\frac{dy}{dx} = \frac{-2x-5}{-8y+3}}$$

Example 2: Let $y = x^2y^3 + y^2x^3$, Find the derivative of this curve $\frac{dy}{dx}$

Solution: Taking the the derivative of both sides we see that,

$$\tfrac{d}{dx}[y] = \tfrac{d}{dx}[x^2y^3 + y^2x^3] \qquad \Rightarrow \qquad \tfrac{dy}{dx} = \tfrac{d}{dx}[x^2y^3] + \tfrac{d}{dx}[y^2x^3]$$

Remember that y is an implicit function of x, so to differentiate the remaining to terms x^2y^3 and y^2x^3, we need to first apply the product rule and then apply the chain rule.

$$\frac{dy}{dx} = [2xy^3 + x^2 3y^2 \frac{dy}{dx}] + [2y\frac{dy}{dx}x^3 + y^2 3x^2]$$

Now we want to solve for $\frac{dy}{dx}$ by isolating every term with a $\frac{dy}{dx}$ on one side, and then factoring out $\frac{dy}{dx}$,

$$\begin{aligned} & \frac{dy}{dx} - x^2 3y^2 \frac{dy}{dx} - 2y\frac{dy}{dx}x^3 = 2xy^3 + y^2 3x^2 \\ \Rightarrow \quad & \frac{dy}{dx}[1 - x^2 3y^2 - 2yx^3] = 2xy^3 + y^2 3x^2 \\ \Rightarrow \quad & \boxed{\frac{dy}{dx} = \frac{2xy^3 + 3y^2x^2}{1 - 3x^2y^2 - 2yx^3}} \end{aligned}$$

Example 3: Recall from chapter 3.2, we proved the exponential rule $\frac{d}{dx}[a^x] = ln(a)a^x$ using the definition of derivative (the limit of the difference quotients). But now that we have the tool of implicit differentiation, we can prove this rule in a much simpler fashion.

So let $y = a^x$ for some constant $a > 0$

To make this equation nicer, let's take the natural log of both sides,

$$ln(y) = ln(a^x) \qquad \Rightarrow \qquad ln(y) = x \cdot ln(a)$$

Now by differentiating both sides we get,

$$\begin{aligned} & \frac{d}{dx}[ln(y)] = \frac{d}{dx}[x \cdot ln(a)] \\ \Rightarrow \quad & \frac{d}{dx}[ln(y)] = ln(a) \end{aligned}$$

Now recall that $\frac{d}{dx}[ln(x)] = \frac{1}{x}$, so using this fact and the chain rule we see that,

$$\frac{1}{y}\frac{dy}{dx} = ln(a)$$

Remember that we originally defined $y = a^x$, so,

$$\begin{aligned} & \frac{1}{a^x}\frac{dy}{dx} = ln(a) \\ \Rightarrow \quad & \boxed{\frac{dy}{dx} = ln(a)a^x} \end{aligned}$$