1. Section 3.6 of HH - The Chain Rule and Inverse Functions

   We know that the graphs of a function and its inverse are reflections of one another across the line $y = x$. Therefore, we can expect that their derivatives will be related as well. In this section we'll show that given a function $y = f(x)$ (that we know how to differentiate) we can use the chain rule to find the derivative of its inverse. In particular, we will show the following:

   - $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$
   - $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{1+x^2}$
   - $\dfrac{d}{dx}(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}}$

   (a) Let's start by finding the derivative of $y = \ln x$. Since the inverse of $\ln x$ is e^x, then $e^y = x$. Differentiate both sides of this equation with respect to x and use the chain rule to find $\frac{dy}{dx}$:

   $$\begin{aligned} &\frac{d}{dx}(e^y) = \frac{d}{dx}(x) \\ &\Rightarrow e^y \cdot \frac{dy}{dx} = 1, \quad \text{by the chain rule} \\ &\Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}, \quad \text{since } e^y = x. \end{aligned}$$

   Since $y = \ln x$ we have shown that $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$.

   (b) Let's use the same procedure to find the derivative of $y = \arctan x$. Since the inverse of $\arctan x$ is $\tan x$, then $\tan y = x$. So by the Pythagorean Identity, $\sec^2 y = 1 + \tan^2 y = 1 + x^2$. Therefore,

$$
\begin{aligned}
&\frac{d}{dx}(\tan y) = \frac{d}{dx}(x) \\
&\Rightarrow \sec^2 y \cdot \frac{dy}{dx} = 1, \quad \text{by the chain rule} \\
&\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+x^2}.
\end{aligned}
$$

(c) Now suppose that f is any invertible function. Let $y = f^{-1}(x)$. Then $f(y) = x$. So, differentiating both sides of this equation with respect to x gives:

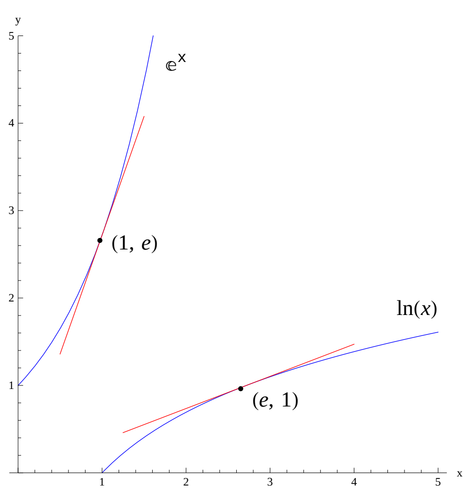
$$
\begin{aligned}
&\frac{d}{dx}(f(y)) = \frac{d}{dx}(x) \\
&\Rightarrow f'(y) \cdot \frac{dy}{dx} = 1, \quad \text{by the chain rule} \\
&\Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.
\end{aligned}
$$

Therefore, we have shown:

If f is an invertible function and $y = f^{-1}(x)$, then

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.$$

Below is a graph of $f(x) = e^x$ and $f^{-1}(x) = \ln x$, along with the tangent lines at the points $(1, e)$ and $(e, 1)$, respectively.



We see that in order to find the derivative of $\ln x$ at the point $(e, 1)$, we invert the value of the derivative of e^x at the point $(1, e)$. In general, the derivative of f^{-1} at a point (a, b) is equal to the reciprocal of the derivative of f at the point (b, a).

(d) We can use the above result to find the derivative of $y = \arcsin x$. Since $\arcsin x$ is the inverse of $\sin x$, then $\sin y = x$. So, by the Pythagorean Identity,

$$\begin{aligned} &\sin^2 y + \cos^2 y = 1 \\ &\Rightarrow \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}. \end{aligned}$$

Therefore, since $\frac{d}{dx}(\sin x) = \cos x$, we have that

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}.$$

2. Examples

(a) Find the derivative of $f(x) = \ln(\sin x)$.

Solution. Since the derivative of $\ln x$ is $\frac{1}{x}$, then by the chain rule,

$$f'(x) = \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) = \frac{1}{\sin x}(\cos x) = \cot x.$$

(b) Find the derivative of $f(x) = \arctan\left(\frac{x}{x+1}\right)$.

Solution. Since the derivative of $\arctan x$ is $\frac{1}{1+x^2}$, then by the chain rule

$$\begin{aligned} f'(x) &= \frac{1}{1 + \left(\frac{x}{x+1}\right)^2} \cdot \frac{d}{dx}\left(\frac{x}{x+1}\right) \\ &= \frac{1}{1 + \frac{x^2}{(x+1)^2}} \cdot \frac{(x+1)\frac{d}{dx}(x) - x \cdot \frac{d}{dx}(x+1)}{(x+1)^2}, \quad \text{by the quotient rule} \\ &= \frac{1}{1 + \frac{x^2}{(x+1)^2}} \cdot \frac{x + 1 - x}{(x+1)^2} \\ &= \frac{1}{1 + \frac{x^2}{(x+1)^2}} \cdot \frac{1}{(x+1)^2} \\ &= \frac{1}{(x+1)^2 + x^2} \\ &= \frac{1}{2x^2 + 2x + 1}. \end{aligned}$$

(c) Find the derivative of $f(x) = \cos(\arcsin x)$.

Solution. By the chain rule,

$$f'(x) = -\sin(\arcsin x) \cdot \frac{d}{dx}(\arcsin x) = -\sin(\arcsin x)\left(\frac{1}{\sqrt{1-x^2}}\right) = \frac{-x}{\sqrt{1-x^2}}.$$

(d) Use the table below, and that fact that f is invertible and differentiable, to find $(f^{-1})'(3)$.

x	$f(x)$	$f'(x)$
3	1	7
6	2	10
9	3	5

Solution. First note that since $f(9) = 3$, then $f^{-1}(3) = 9$. So, in order to compute the value of the derivative of f^{-1} at the point $(3,9)$, we need to find the reciprocal of the value of the derivative of f at the point $(9,3)$. That is,

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(9)} = \frac{1}{5}.$$