1. Section 3.5 of HH - Trigonometric Functions

   In this section we will find the derivatives of the following trigonometric functions:

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$

(a) First, let's try drawing a graph of the derivative of $\sin x$. Below is a graph of $y = \sin x$, along with the tangent lines at $x = 0, \pi$, and 2π.

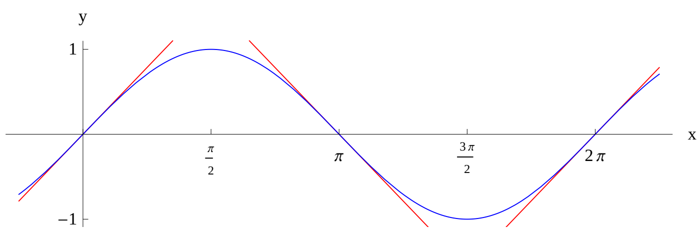

Figure 1: $y = \sin x$

We see that the derivative is zero at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. Moreover, it looks like the derivative has a maximum value of 1 when $x = 0$ and 2π, and a minimum value of -1 when $x = \pi$.

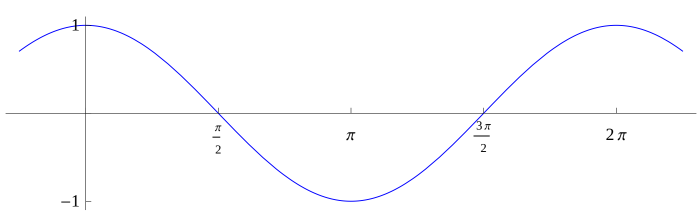

Figure 2: derivative of $y = \sin x$

This looks a lot like the graph of $y = \cos x$. It turns out, it is! From now on we will assume that $\frac{d}{dx}(\sin x) = \cos x$. You can find an informal geometric proof of this fact on pages 142-143 of the text.

Warning! The above result is only valid when x is radians.

(b) We saw in section 1.5 that $\cos x = \sin(x + \frac{\pi}{2})$. Therefore, by the chain rule

$$\begin{aligned}\frac{d}{dx}(\cos x) &= \frac{d}{dx}(\sin(x + \frac{\pi}{2})) \\ &= \cos(x + \frac{\pi}{2}) \\ &= \sin(x + \pi) \\ &= -\sin x.\end{aligned}$$

(c) Now we can find the derivative of $\tan x$.

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{(\cos x) \cdot \frac{d}{dx}(\sin x) - (\sin x) \cdot \frac{d}{dx}(\cos x)}{\cos^2 x}, \quad \text{by the quotient rule} \\ &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x.\end{aligned}$$

(d) Last, let's find the derivative of $\sec x$.

$$\begin{aligned}\frac{d}{dx}(\sec x) &= \frac{d}{dx}((\cos x)^{-1}) \\ &= -(\cos x)^{-2} \cdot \frac{d}{dx}(\cos x), \quad \text{by the chain rule} \\ &= -(\cos x)^{-2}(-\sin x) \\ &= \frac{\sin x}{\cos^2 x} \\ &= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \\ &= \sec x \tan x.\end{aligned}$$

The derivatives of the other trigonometric functions, $\csc x$ and $\cot x$, can be found similarly.

2. Examples

(a) Find the derivative of $f(x) = \sec(3x)$.

Solution. By the chain rule,

$$f'(x) = \sec(3x)\tan(3x) \cdot \frac{d}{dx}(3x) = 3\sec(3x)\tan(3x).$$

(b) Find the derivative of $f(x) = \sin x \cos x$.

Solution. By the product rule,

$$\begin{aligned} f'(x) &= (\sin x) \cdot \frac{d}{dx}(\cos x) + (\cos x) \cdot \frac{d}{dx}(\sin x) \\ &= \sin x(-\sin x) + \cos x \cos x \\ &= \cos^2 x - \sin^2 x. \end{aligned}$$

(c) Find the derivative of $f(x) = \sin(\cos x)$.

Solution. By the chain rule,

$$f'(x) = \cos(\cos x) \cdot \frac{d}{dx}(\cos x) = \cos(\cos x)(-\sin x) = -\sin x \cos(\cos x).$$

(d) Find the derivative of $f(x) = xe^{\tan x}$

Solution. By the product and chain rules,

$$\begin{aligned} f'(x) &= x \cdot \frac{d}{dx}(e^{\tan x}) + (e^{\tan x}) \cdot \frac{d}{dx}(x) \\ &= x(e^{\tan x} \cdot \frac{d}{dx}(\tan x)) + (e^{\tan x})(1) \\ &= x(e^{\tan x}\sec^2 x) + e^{\tan x} \\ &= e^{\tan x}(x\sec^2 x + 1). \end{aligned}$$