Math 1300: Calculus 1
Section 3.4: Chain Rule
October 4, 2013

Example 1. *Suppose you are traveling by airplane. The function $H = f(t)$ gives the altitude with respect to time. Also, the function $T = g(a)$ gives temperature with respect to altitude. How does temperature change with respect to time?*

Temperature is affected by rate of change in altitude and how fast the plane is ascending. We can think of Temperature as a composite function, $T = g \circ f(t)$ with respect to time. This suggests that the rate of change of the composite function is the product of the rate of change of the outside function, g, and the inside function f.

Theorem 1 (Chain Rule)**.** *If f and g are differentiable functions, then*

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Example 2. *Let $f(x) = x^2$ and $g(x) = x^{-1}$ find $\frac{d}{dx}[f \circ g(x)]$. Then verify answer with the chain rule.*

First we know that $f \circ g(x) = (x^{-1})^2 = x^{-2}$ and by using the power rule $\frac{d}{dx}[f \circ g(x)] = -2x^{-3}$. Now by chain rule we have that the outer function is f and the inner function is g. Therefore $f'(g(x)) = 2(x^{-1})$ and $g'(x) = -x^{-2}$ so by using the chain rule we get

$$\frac{d}{dx}[f \circ g(x)] = f'(g(x))g'(x) = 2(x^{-1})(-x^{-2}) = -2x^{-3}.$$

Example 3. *Using the chain rule we can find the derivative to $(2 - 4x^2)^{100}$. First notice this is a difficult expression to expand and without the chain rule taking this derivative would be miserable. We identify the outer function $f(x) = x^{100}$ and the inner function as $g(x) = 2 - 4x^2$. That is $f \circ g(x) = (2 - 4x^2)^{100}$. We first calculate $f'(g(x)) = 100(2 - 4x^2)^{99}$ and $g'(x) = -8x$; therefore by the chain rule,*

$$\frac{d}{dx}\left((2 - 4x^2)^{100}\right) = \frac{d}{dx}[f \circ g(x)] = f'(g(x))g'(x) = \left(100(2 - 4x^2)^{99}\right)(-8x) = -800x(2 - 4x^2)^2.$$

Example 4. *We can also apply the chain rule to the case of exponentials, for example find $\frac{d}{dx} e^{e^x}$. We define $f(x) = e^x$ and $g(x) = e^x$ then $f'(g(x)) = e^{e^x}$ and $g'(x) = e^x$. Therefore by applying the chain rule,*

$$\frac{d}{dx}\left(e^{e^x}\right) = \frac{d}{dx}[f \circ g(x)] = f'(g(x))g'(x) = e^{e^x} e^x = e^{e^x + x}.$$

Example 5. *We can derive the quotient rule a new way by applying both the chain rule and the product rule.*

$$\begin{aligned}\frac{d}{dx}\left(\frac{p(x)}{q(x)}\right) &= \frac{d}{dx}\left(p(x)q(x)^{-1}\right)\\ &= p'(x)q(x)^{-1} + p(x)(-q(x)^{-2}q'(x))\\ &= \frac{p'(x)q(x) - p(x)q'(x)}{(q(x))^2}\end{aligned}$$

Example 6. *Find*

$$\frac{d}{dx}\sqrt{x + 2^x}$$

We first identify the outer function $f(x) = x^{\frac{1}{2}}$ and then inside function as $g(x) = x+2^x$. Then $f'(g(x)) = \frac{1}{2\sqrt{x+2^x}}$ and $g'(x) = 1 + 2^x \ln(2)$. Then by applying the chain rule we get,

$$\frac{d}{dx}\left(\sqrt{x + 2^x}\right) = \frac{d}{dx}[f \circ g(x)] = f'(g(x))g'(x) = \frac{1 + 2^x \ln(2)}{2\sqrt{x + 2^x}}.$$

In the next example we can see how in some cases we need a combination of both the chain and product rule.

Example 7. *Find* $\frac{d}{dx}\left(x \cdot e^{x^3+7x}\right)$

First we write $x \cdot e^{x^3+7x} = h_1(x)h_2(x)$ *where* $h_1(x) = x$ *and* $h_2(x) = e^{x^3+7x}$*. We know that* $h_1'(x) = 1$ *but we need the chain rule to find* $h_2'(x)$*. We decompose* $h_2(x) = f(g(x))$ *where* $f(x) = e^x$ *and* $g(x) = x^3 + 7x$ *therefore we find* $f'(g(x)) = e^{x^3+7x}$ *and* $g'(x) = 3x^2 + 7$*.*

$$h_2'(x) = f'(g(x))g'(x) = (3x^2+7)e^{3x^2+7}$$

Now we are ready to apply the product rule

$$\begin{aligned}\frac{d}{dx}\left(x \cdot e^{x^3+7x}\right) &= \frac{d}{dx}h_1(x)h_2(x)\\ &= h_1'(x)h_2(x) + h_1(x)h_2'(x)\\ &= e^{x^3+7x} + x(3x^2+7)e^{3x^2+7}\\ &= (1+3x^3+7x)e^{x^3+7x}\end{aligned}$$

It is often the case where we are challenged with the task of differentiating a function which is the composition of three functions. In this case we need to recursively apply the chain rule. The following proposition will be useful.

Proposition 1. *We differentiating a function which is the composition of three functions,*

$$\frac{d}{dx}f(g(h(x))) = f'(g(h(x))) \cdot g'(h(x)) \cdot h(x)$$

Proof: First let $P(x) = g(h(x))$. Then $f(g(h(x))) = f(P(x))$ and

$$\frac{d}{dx}f(P(x)) = f'(P(x)) \cdot P'(x).$$

Notice that $P'(x) = \frac{d}{dx}g(h(x)) = g'(h(x)) \cdot h'(x)$, then by substitution we obtain

$$\frac{d}{dx}f(g(h(x))) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

□

Example 8. *Find* $\frac{d}{dx}2^{\sqrt{x^{-1}-x}}$

Here we have a composition of three functions where $f(x) = 2^x$*,* $g(x) = \sqrt{x}$*, and* $h(x) = x^{-1} - x$ *where* $2^{\sqrt{x^{-1}-x}} = f(g(h(x)))$*. We first find the following*

$$\begin{aligned}f'(g(h(x))) &= \ln(2)2^{\sqrt{x^{-1}-x}}\\ g'(h(x)) &= \frac{1}{2\sqrt{x^{-1}-x}}\\ h'(x) &= \frac{-1}{x^2} - 1\end{aligned}$$

and now we are ready to apply the above proposition,

$$\begin{aligned}\frac{d}{dx}2^{\sqrt{x^{-1}-x}} &= \frac{d}{dx}f(g(h(x)))\\ &= f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)\\ &= \ln(2)2^{\sqrt{x^{-1}-x}} \cdot \frac{1}{2\sqrt{x^{-1}-x}} \cdot \left(\frac{-1}{x^2} - 1\right).\end{aligned}$$

Example 9. *Find* $\frac{d}{dx}\sqrt{x+\sqrt{x+\sqrt{x}}}$

Here we have a composition of three functions where $f(x) = \sqrt{x}$ *and* $g(x) = x + \sqrt{x+\sqrt{x}}$*. To differentiate* $f \circ g(x)$ *we first need to find* $g'(x)$ *to do this we write* $g(x) = x + h_1(h_2(x))$ *where* $h_1(x) = \sqrt{x}$ *and* $h_2(x) = x + \sqrt{x}$ *from this we have that*

$$g'(x) = 1 + h_1'(h_2(x))h_2'(x) = 1 + \frac{1}{2\sqrt{x+\sqrt{x}}}\left(1 + \frac{1}{2\sqrt{x}}\right).$$

The next step is to find $f'(g(x))$*,*

$$f'(g(x)) = \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}.$$

Finally we calculate

$$\frac{d}{dx}\sqrt{x+\sqrt{x+\sqrt{x}}} = f'(g(x))g'(x) = \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left(\left(1 + \frac{1}{2\sqrt{x+\sqrt{x}}}\right)\left(1 + \frac{1}{2\sqrt{x}}\right)\right).$$