MATH 1300 Lecture Notes for Section 3.3 - the product and quotient rules

The purpose of this section is to learn how to take the derivative of a product and of a quotient.

Products

How NOT to do it: Many students guess that if you have a product of two functions $f(x)g(x)$, that you should be able to take the derivative of the functions separately, and then multiply the derivatives to get $f'(x)g'(x)$. This "fake product rule" DOES NOT work. It DOES NOT give the correct derivative.

Bottom line, The Product Rule:

$$(fg)' = f'g + g'f$$

You should memorize it in words, too: the derivative of a product is the derivative of the first times the second, plus the derivative of the second times the first.

Example: Find the derivative of $h(x) = x^2e^x$
This is a product $f(x)g(x)$ where $f(x) = x^2$ and $g(x) = e^x$. $f'(x) = 2x$ and $g'(x) = e^x$.
By the product rule $h'(x) = \frac{d}{dx}(x^2e^x) = f'g + g'f = 2xe^x + e^x \cdot x^2 = 2xe^x + x^2e^x$.

Quotients

Bottom line, The Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

In words: The derivative of a quotient is bottom times the derivative of the top minus the top times the derivative of the bottom, all over the bottom squared.

Example: Find the derivative of $h(x) = \frac{x^2+5x}{3x+1}$
Here the "top" function is $f(x) = x^2 + 5x$ and the "bottom" function is $g(x) = 3x + 1$. The derivative of the top is $f'(x) = 2x + 5$ and the derivative of the bottom is $g'(x) = 3$.

$$h'(x) = \frac{d}{dx}\left(\frac{x^2+5x}{3x+1}\right) = \frac{gf' - fg'}{g^2} = \frac{(3x+1)(2x+5) - (x^2+5x)(3)}{(3x+1)^2}$$

This can be simplified of course:

$$= \frac{6x^2 + 17x + 5 - 3x^2 - 15x}{(3x+1)^2} = \frac{3x^2 + 2x + 5}{(3x+1)^2}$$

More complicated example: Find the derivative of $f(x) = \frac{(4x^3+7)\cdot e^x}{x^2+1}$
Notice that this example has a product in the numerator of a quotient. We must use the quotient rule, and in the middle of it, when we get to the part where we take the derivative of the top, we must use a product rule to calculate that. It might stretch your brain to keep track of where you are in this process. When you complete using the product rule in the middle of this calculation, you have to return to the quotient rule, remembering what step you were on. Pay attention and practice this until you get really good, because in upcoming sections it will get even more complicated.

$$\begin{aligned} f'(x) &= \frac{(x^2+1)\frac{d}{dx}((4x^3+7)\cdot e^x) - (4x^3+7)\cdot e^x \cdot \frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\ &= \frac{(x^2+1)\cdot(12x^2e^x + e^x(4x^3+7)) - (4x^3+7)e^x 2x}{(x^2+1)^2} \\ &= \frac{(x^2+1)(4x^3+12x^2+7)e^x - (8x^4+14x)e^x}{(x^2+1)^2} \end{aligned}$$

This expression can be simplified further, but typically we hold on off on the simplifying unless we need to do it as part of a larger problem.

Example: before taking derivatives, it is usually helpful to simplify first:
Find the derivative of $f(x) = \frac{x^2+1}{\sqrt{x}}$
This is a quotient, so you could certainly use the quotient rule to find this derivative. But it will be much easier to simplify it algebraically first, and then just use the rules of section 3.1:
$f(x) = \frac{x^2+1}{\sqrt{x}} = \frac{x^2}{\sqrt{x}} + \frac{1}{\sqrt{x}} = x^{3/2} + x^{-1/2}$
$f'(x) = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-3/2} = \frac{3}{2}\sqrt{x} - \frac{1}{2x^{3/2}}$

Numerical example:
Say that $h(x) = f(x)g(x)$. Say that $f'(1) = 2$, $f(1) = 3$, $g(1) = 4$, and $g'(1) = 5$. What is $h(1)$ and $h'(1)$?
$h(1)$ is just defined as a product: $h(1) = f(1)g(1) = 3 \cdot 4 = 12$.
But $h'(1)$ has to be calculated by the product rule for derivatives.

$$h'(1) = f'(1)\cdot g(1) + g'(1)\cdot f(1) = 2\cdot 4 + 5\cdot 3 = 8 + 15 = 23$$

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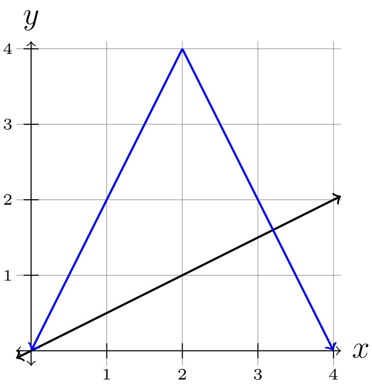
Graphical example:
In the graph below, $f(x)$ is shown in black and $g(x)$ is shown in blue:



Suppose that $h(x) = f(x)g(x)$. Find $h(3)$ and $h'(3)$
Solution: $h(x)$ is just a product, so $h(3) = f(3)g(3) = \frac{3}{2} \cdot 2 = 3$.
But $h'(x)$ must be calculated by the product rule.

$$h'(x) = f'(x)g(x) + g'(x)f(x)$$

$$h'(3) = f'(3)g(3) + g'(3)f(3)$$

The values of $f'(3)$ and $g'(3)$ can be found from the graph. $f'(3)$ is the slope of f at $x = 3$, which is $\frac{1}{2}$. $g'(3)$ is the slope of g at $x = 3$, which is -2.

$$h'(3) = f'(3)g(3) + g'(3)f(3) = \frac{1}{2} \cdot 2 + -2 \cdot \frac{3}{2} = 1 - 3 = -2$$