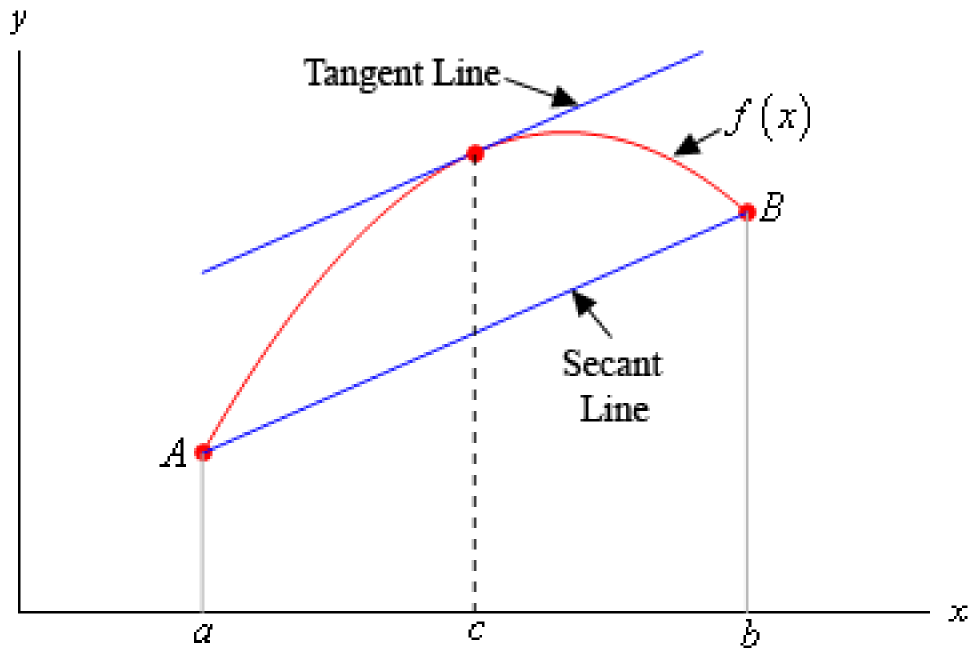
MATH 1300 Lecture Notes for Section 3.10 - Theorems about differentiable functions

Theorems you should know about differentiable functions:

- The Mean Value Theorem
- The Increasing Function Theorem/The Decreasing Function Theorem
- The Constant Function Theorem

The Mean Value Theorem
The main importance of the Mean Value Theorem is that it is necessary to prove other very useful theorems.
In words: The Mean Value Theorem (MVT) roughly says that if you have a differentiable function, then the slope of the secant line through two points on the function will be matched by the slope of the tangent line somewhere in-between.

Graphically:



Analytically:
If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is a value c in the interval (a, b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The *hypotheses* of the MVT are the assumptions: that the function is continuous on the closed interval and differentiable on the open interval. The *conclusion* is that there is an x-value ($x = c$) where the slope of the tangent equals the slope of the secant (that is, $f'(c) = \frac{f(b)-f(a)}{b-a}$).

If the hypotheses are not satisfied, then the MVT does not apply, and the conclusion might or might not be satisfied, it depends on the situation. But if the hypotheses *are* satisfied, then the theorem *guarantees* that the conclusion will be met, too.

Example: Consider the function $f(x) = x^2 - 2x$ on the interval $[0, 3]$. Are the hypotheses of the MVT satisfied? Are the conclusions of the MVT satisfied?
Solution: $f(x)$ is a polynomial, so it is continuous everywhere (and in particular on the interval $[0, 3]$), and differentiable everywhere (in particular on the interval $(0, 3)$). So the hypotheses of the MVT are satisfied. Therefore, the conclusion of the MVT *must* be satisfied, because the theorem guarantees it. The theorem *does not* tell me anything about how to find the value of c, but in this case I can find it directly. First, the slope of the secant line is $\frac{f(3)-f(0)}{3-0} = \frac{9-6}{3} = 1$. I want to find c so that $f'(c) = 1$. $f'(x) = 2x - 2 = 1$. I need to solve $f'(c) = 2c - 2 = 1$, which gives $c = 3/2$.
There are not a lot of practical uses for finding the value of c. The whole point of this exercise is to illustrate and practice the meaning of the MVT, so we can confidently use it to prove other theorems.

Example: For $f(x) = |x|$ on the interval $[-2, 2]$, do the hypotheses and the conclusion of the MVT hold?
Solution: $f(x)$ has a cusp at $x = 0$, which is inside the interval, so the hypotheses of the MVT do not hold. The theorem does not apply, so the conclusion might or might not hold. To find out if the conclusion happens to hold, we have to check directly. The slope of the secant line $\frac{f(2)-f(-2)}{2-(-2)} = 0$. The derivative of $f(x)$ is never 0, so the conclusion of the MVT is not satisfied.

The Increasing Function Theorem
Suppose that f is continuous on $a \leq x \leq b$ and differentiable on $a < x < b$.

- If $f'(x) > 0$ on $a < x < b$, then f is increasing on $a \leq x \leq b$.
- If $f'(x) \geq 0$ on $a < x < b$, then f is nondecreasing on $a \leq x \leq b$.

The Increasing Function Theorem has a twin:
The Decreasing Function Theorem
Suppose that f is continuous on $a \leq x \leq b$ and differentiable on $a < x < b$.

- If $f'(x) < 0$ on $a < x < b$, then f is decreasing on $a \leq x \leq b$.
- If $f'(x) \leq 0$ on $a < x < b$, then f is nonincreasing on $a \leq x \leq b$.

The Increasing Function Theorem has a cousin:
The Constant Function Theorem
Suppose that f is continuous on $a \leq x \leq b$ and differentiable on $a < x < b$. If $f'(x) = 0$ on a $< x < b$, then f is constant on $a \leq x \leq b$.

Though it seems like these theorems should be obvious, their proofs (which you may read in the textbook) are not trivial because the MVT is required to prove them. These theorems (among other future important and useful theorems) are the reason the MVT is important.

Example: Show that the function $f(x) = x^3 + x$ is increasing everywhere.
Solution: $f(x)$ is a polynomial, so it is continuous and differentiable for all real numbers. $f'(x) = 3x^2 + 1 > 0$, so by the Increasing Function Theorem, $f(x)$ is increasing for all real numbers. Note: Functions that are increasing everywhere are invertible. So $f(x)$ is an invertible function. It is hard to find a formula for its inverse, though.