MATH 1300 **Lecture Notes** **Wednesday, September 25, 2013**

1. Section 3.1 of HH - Powers and Polynomials

   In this section 3.1, you are given several differentiation rules that, taken altogether, allow you to quickly and easily differentiate all power functions and all polynomial functions.

   Recall that power functions have the form $f(x) = kx^p$, where k and p are any constants. Examples of power functions are $f(x) = 2x^3$, $g(x) = 3x^{-2}$, and $h(x) = 2x^{\frac{2}{3}}$

   Polynomial functions are sums and differences of power functions in which the power p is restricted to be non-negative integers $0, 1, 2, 3$, etc..

   An example of a polynomial function is $p(x) = 2x^3 - 5x^2 + 6x - 11$

   Now here is an itemized list of the promised differentiation rules that, taken altogether, allow you to quickly and easily differentiate all power functions and all polynomial functions.

   (a) The Derivative of a Constant Multiple of a function

   When you differentiate $cf(x)$ with c being any constant, the c can come out in front of the result of differentiating the $f(x)$. In symbols:

   $\dfrac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$

   This is Theorem 3.1 on page 116 in your textbook. The proof of this theorem using the limit of a difference quotient immediately follows the statement of the theorem on page 116.

   In words, we say that multiplicative constants can come out in front of derivatives.

   See section 2(a) in the Examples-section below for some examples of the use of this rule.

   (b) The Derivative of a Sum and Difference of functions

   When you differentiate $f(x) + g(x)$, you can differentiate each of $f(x)$ and $g(x)$ separately and then add the derivatives together.

   Likewise, when you differentiate $f(x) - g(x)$, you can differentiate each of $f(x)$ and $g(x)$ separately and then subtract the derivatives.

   In symbols: $\dfrac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

   In words, the derivative of a sum is the sum of the derivatives.

   $\dfrac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

   In words, the derivative of a difference is the difference of the derivatives.

   Sometimes you will see both of these differentiation rules written as one:

$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
where you use the top sign on both sides or the bottom sign on both sides.

This is Theorem 3.2 on page 117 in your textbook. The proof of this theorem using the limit of a difference quotient immediately follows the statement of the theorem on page 117.

See section 2(b) in the Examples-section below for some examples of the use of these rules.

(c) The Power Rule

The Power Rule allows you to differentiate any power function. In symbols:
$\frac{d}{dx}[x^n] = nx^{n-1}$ where n is any constant.

The Power Rule is stated at the top of page 118 in your textbook, and the justification for the Power Rule for positive integers n is shown at the top of page 119. The justification for the Power Rule for other kinds of constants n like negative integer constants, fractions, and indeed all real number constants will occur later in the course. Watch for these justifications as the course progresses.

Three examples of the use of the Power Rule will be given here, for use in the early parts of the Examples section below. More examples of the Power Rule will be given in the Power Rule section in the Examples below.

With $n = 2$, $\frac{d}{dx}[x^2] = 2x^{2-1} = 2x^1 = 2x$

With $n = 3$, $\frac{d}{dx}[x^3] = 3x^{3-1} = 3x^2$

With $n = 4$, $\frac{d}{dx}[x^4] = 4x^{4-1} = 4x^3$

See Examples 1 and 2 on page 118 in your textbook for examples of the use of the Power Rule. Also see section 2(c) in the Examples-section below for more examples of the use of the Power Rule.

(d) Polynomials

We now have enough differentiation rules to be able to differentiate any polynomial function. Recall that a polynomial function is a sum and/or difference of certain power functions. We can now pull apart the sums and differences (using The Derivative of a Sum and Difference of functions) and differentiate the resulting individual terms term-by-term, then pull off any constant in front of each power function term (using The Derivative of a Constant Multiple of a function), and then differentiate the bare-bones power function using the Power Rule.

A completely worked-out example of the differentiation of a polynomial function occurs in the Examples section below in 2(d).

(e) The Second Derivative

You learned about the second derivative back in section 2.5 in your textbook. Now we have the tools to be able to compute the second derivatives of power functions and polynomial functions easily and quickly, because the derivative of a power function is a power function (and so can be easily differentiated a second time), and the derivative of a polynomial function is a polynomial function (and so can be easily differentiated a second time). But the interpretation of the sign of the second derivative continues to hold: the graph of a function $y = f(x)$ is concave-up (CU) on any interval over which $f''(x) > 0$ for all x in that interval, and the graph of a function $y = f(x)$ is concave-down (CD) on any interval over which $f''(x) < 0$ for all x in that interval.

2. Examples

(a) The Derivative of a Constant Multiple of a function

The first example here uses $\dfrac{d}{dx}[x^2] = 2x$ which was demonstrated above in the section on the Power Rule.

$\dfrac{d}{dx}[3x^2] = 3 \cdot \dfrac{d}{dx}[x^2] = 3(2x) = 6x$

The second example here uses $\dfrac{d}{dx}[x^3] = 3x^2$ which was demonstrated above in the section on the Power Rule.

$\dfrac{d}{dx}[-7x^3] = -7 \cdot \dfrac{d}{dx}[x^3] = -7(3x^2) = -21x^2$

In words, we say that multiplicative constants can come out in front of derivatives.

(b) The Derivative of a Sum and Difference of functions

The examples here use the differentiation rules demonstrated above in the section on the Power Rule. The second example here also uses the differentiation rule that allows us to bring multiplicative constants out in front of derivatives.

$\dfrac{d}{dx}[x^4 + x^2] = \dfrac{d}{dx}[x^4] + \dfrac{d}{dx}[x^2] = 4x^3 + 2x$

$\dfrac{d}{dx}[9x^3 - 5x^2 + 6x^4]$

$= \dfrac{d}{dx}[9x^3] - \dfrac{d}{dx}[5x^2] + \dfrac{d}{dx}[6x^4]$

$= 9\dfrac{d}{dx}[x^3] - 5\dfrac{d}{dx}[x^2] + 6\dfrac{d}{dx}[x^4]$

$= 9(3x^2) - 5(2x) + 6(4x^3)$

$= 27x^2 - 10x + 24x^3$

This last example shows that the rules for the derivatives of sums and differences of functions are not limited to just sums and/or differences of two functions but rather sums and/or differences of any number of functions.

In words, we say that the derivative of a sum is the sum of the derivatives. And, the derivative of a difference is the difference of the derivatives.

(c) The Power Rule

With $n = 100$, $\dfrac{d}{dx}[x^{100}] = 100x^{100-1} = 100x^{99}$

With $n = 1$, $\dfrac{d}{dx}[x] = \dfrac{d}{dx}[x^1] = 1x^{1-1} = x^0 = 1$

With $n = 0$, $\dfrac{d}{dx}[1] = \dfrac{d}{dx}[x^0] = 0x^{0-1} = 0$

With $n = -1$, $\dfrac{d}{dx}\left[\dfrac{1}{x}\right] = \dfrac{d}{dx}[x^{-1}] = -1x^{-1-1} = -x^{-2} = -\dfrac{1}{x^2}$

With $n = \dfrac{4}{5}$, $\dfrac{d}{dx}[\sqrt[5]{x^4}] = \dfrac{d}{dx}[x^{\frac{4}{5}}] = \dfrac{4}{5}x^{\frac{4}{5}-1} = \dfrac{4}{5}x^{-\frac{1}{5}} = \dfrac{4}{5} \cdot \dfrac{1}{x^{\frac{1}{5}}} = \dfrac{4}{5} \cdot \dfrac{1}{\sqrt[5]{x}} = \dfrac{4}{5\sqrt[5]{x}}$

Look carefully at the three examples given in Example 1 on page 118 in your textbook. That Example 1 parts (b) and (c) give you more examples about how to use the Power Rule when the constant exponent n is a fraction, either positive or negative.

(d) Polynomials

Here's a completely worked-out example, which you should strive to eventually be able to accomplish in one step.

$$\frac{d}{dx}[5x^3 - 8x^2 + 3x - 6]$$

$$= \frac{d}{dx}[5x^3] - \frac{d}{dx}[8x^2] + \frac{d}{dx}[3x] - \frac{d}{dx}[6]$$ [Derivative of sum and difference]

$$= 5\frac{d}{dx}[x^3] - 8\frac{d}{dx}[x^2] + 3\frac{d}{dx}[x] - 0$$ [Multiplicative constants come out in front of derivatives, and for the last term: the derivative of any constant is 0]

$$= 5(3x^2) - 8(2x) + 3(1)$$ [Power Rule, applied three times]

$$= 15x^2 - 16x + 3$$

You can find four more completely worked-out examples in your textbook: see Example 3 at the bottom of page 119, and Example 4 near the top of page 120.

(e) The Second Derivative

Example 5 starting in the middle of page 120 in your textbook shows three examples of computing second derivatives and then interpreting their signs to give information about the concavity of the original function.

Example 6 starting at the very bottom of page 120 differentiates a distance function twice, the first derivative yielding the velocity function and the second derivative yielding the acceleration function. The first derivative of a distance function is a velocity function because velocity is the rate at which distance changes in time. In turn, the rate at which the velocity changes in time is what we call the acceleration. So the acceleration function is the second derivative of the distance function.