Section 2.6 - Differentiability

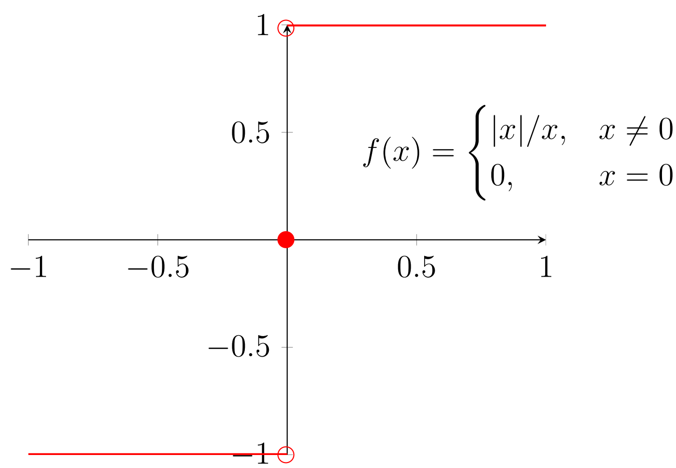
- *Definition:* A function f is called <u>differentiable at x</u> if

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

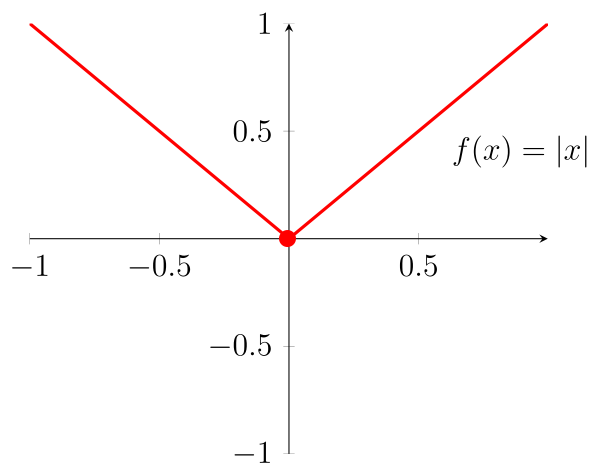
  exists.

- *Note:* Graphically, this means f has a well-defined, non-vertical tangent line (i.e., the graph is smooth) at x.

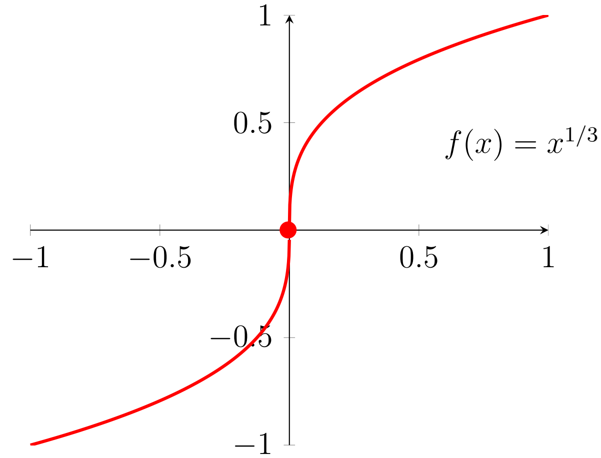
- *When is a Function Not Differentiable?*
  - If f is not continuous at f.



  - If f has a sharp corner (also called a <u>cusp</u>) at x.



- If f has a vertical tangent line at x (since the derivative is the slope of the tangent line and the slope of a vertical line is undefined).



- *Theorem:* If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

- *Note:* Another way of stating the the theorem is: if $f(x)$ is not continuous at $x = a$, then $f(x)$ is not differentiable at $x = a$. In short, all differentiable functions are continuous, but not all continuous functions are differentiable.

- *Example:* Is
$$f(x) = \begin{cases} x, & x < 1 \\ x^2, & x \geq 1 \end{cases}$$
differentiable at $x = 1$?
Solution: We first check if $f(x)$ is continuous at $x = 1$. If it isn't continuous, the theorem says it can't be differentiable, in which case we don't even have to try to compute the derivative. Since $\lim_{x \to 1^-} f(x) = f(1) = \lim_{x \to 1^+} f(x)$, $f(x)$ is continuous at $x = 1$. Because $f(x)$ is continuous at $x = 1$, the theorem does not apply, and we will have to try to compute the derivative from the limit definition to see if the function is differentiable:
$$\begin{aligned} \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0^+} \frac{(1+h)^2 - (1)^2}{h} \\ &= \lim_{h \to 0^+} \frac{1 + 2h + h^1 - 1}{h} \\ &= \lim_{h \to 0^+} \frac{2h + h^2}{h} \\ &= \lim_{h \to 0^+} 2 + h \\ &= 2 \end{aligned}$$

and

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0^-} \frac{(1+h)-(1)}{h} \\
&= \lim_{h\to 0^+} \frac{h}{h} \\
&= \lim_{h\to 0^+} 1 \\
&= 1.
\end{aligned}$$

Since the two limits don't agree, $f(x)$ is not differentiable at $x = 1$. So, $f(x)$ is continuous at $x = 1$, but it is not differentiable at $x = 1$. Notice that this means that the theorem above doesn't work the other way: in general, continuity **does not** imply differentiability. We can see the cusp at $x = 1$ on the graph below.