Sec. 2.5: The Second Derivative

From previous sections, we know that if we have a differentiable function $f(x)$, we can find a derivative function $f'(x)$. But $f'(x)$ is a function itself (possibly differentiable), which raises the question what happens when we take the derivative of the derivative $f'(x)$? For a function f, the derivative of its derivative is called the *second derivative*, and written $f''(x)$ (read"f double-prime".)

Notation: For a function $y = f(x)$ the second derivative can be written in the following two ways,

$$f''(x) = \frac{d^2y}{dx^2}$$

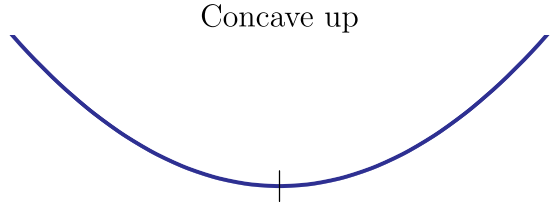
So now you might be wondering, why do we care about the second derivative? what information can we infer about a function f, given information about its second derivative? Well first recall:
· if $f' > 0$ on an interval, then f is increasing over that interval.
· if $f' < 0$ on an interval, then f is decreasing over that interval.

And remember, f'' is simply the derivative of the first derivative of f', so we know:
· if $f'' > 0$ on an interval, then f' is increasing over that interval.
· if $f'' < 0$ on an interval, then f' is decreasing over that interval.

But what does f' increasing or decreasing on an interval tell us about the graph of f? To answer this question we have to first review the concept of concavity.

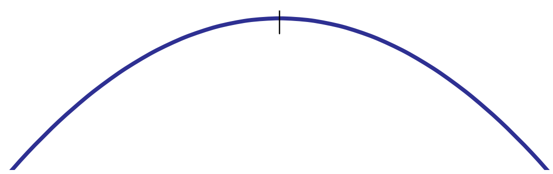
· When a curve bends upwards like in the example below, we call that curve *concave up*.

$f'' > 0$
Concave up



Now if we analyze this curve, you'll notice that to the right of the center, our curve is increasing, so we have $f' > 0$ to the right of the center. But notice that the curve is getting steeper and steeper, which means the function is increasing at a faster and faster rate. This tells us that f' is increasing to the right of the center. Similarly if we look to the left of the center, our curve is decreasing, so we have $f' < 0$ to the left of the center. Now the slope of the curve is getting less steep as we approach the center, so to the far left of the center the rate of change would be a large negative number, and closer to the center the rate of change would be a smaller negative number. So the rate of change is still an increasing quantity, and f' is also increasing to the left of the center. Putting these two pieces together we see that f' is increasing throughout this entire curve, So the derivative of f' is positive, i.e. $f'' > 0$

· When a curve bends downwards like in the example below, we call that curve *concave down*.



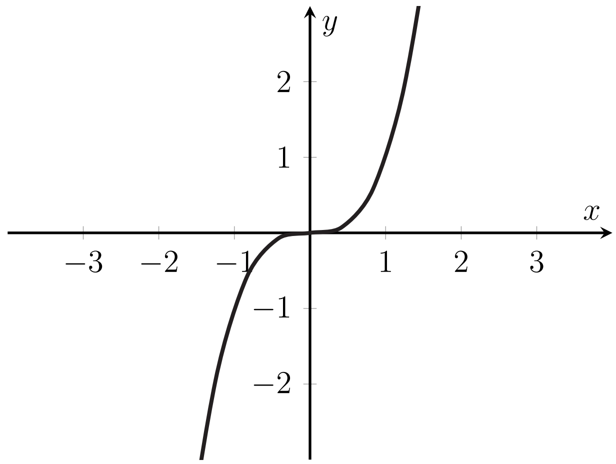
$f'' < 0$
Concave down

Now if we analyze this curve, you'll notice that to the left of the center, our curve is increasing, so we have $f' > 0$ to the left of the center. But notice that the curve is getting more shallow as we approach the center, which means the function is increasing at a slower and slower rate. This tells us that f' is decreasing to the left of the center. Similarly if we look to the right of the center, our curve is decreasing, so we have $f' < 0$ to the right of the center. Now the slope of the curve is getting more steep as we get further away from the center, so to the far right of the center the rate of change would be a large negative number, and closer to the center the rate of change would be a smaller negative number. So the rate of change is still a decreasing quantity, and f' is also decreasing to the right of the center. Putting these two pieces together we see that f' is decreasing throughout this entire curve, so the derivative of f' is negative, i.e. $f'' < 0$

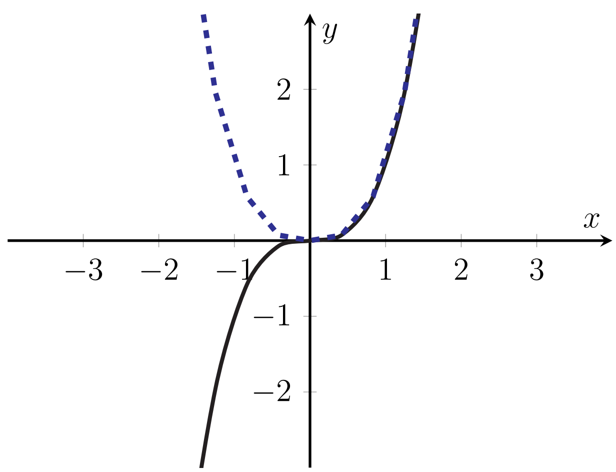
Using the information we just deduced, we get the following result:
· if $f'' > 0$ on an interval, then f' is increasing, so the graph of f is concave up on that interval.
· if $f'' < 0$ on an interval, then f' is decreasing, so the graph of f is concave down on that interval.

*A trick to finding the concavity of a function f on an interval is; if f looks like part of a smiley face on an interval, then f is concave up on that interval. And if f looks like part of a frowny face on an interval, then f is concave down on that interval.

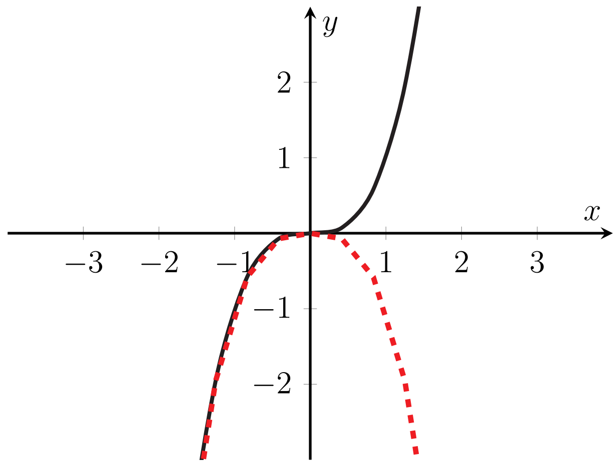
Example 1: Let's say we are given the following graph of a function f and asked to find any and all intervals where the graph is concave up.



You should notice that to the right of the y-axis, the graph looks like part of the smiley face drawn in blue.



This tells us that f is concave up on the interval $(0, \infty)$. Similarly, to the left of the y-axis, the graph looks like part of the frowny face drawn in red.



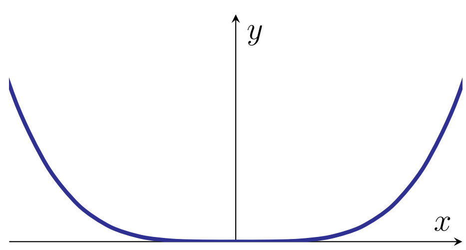
This tells us that f is concave down on the interval $(-\infty, 0)$. Therefore the only interval where f is concave up, is $(0, \infty)$.

Note:

· One thing you may notice from the above example is that at $x = 0$, the graphs concavity switches from down to up. This is what we call an ***inflection point***, which is defined as a point in which the graph of a function changes concavity. One way we identify an inflection point is if a function f has a second derivative f'', then at an inflection point z we must have either $f''(z) = 0$ or $f''(z) = undefined$. And in the above example, it was the case that $f''(0) = 0$.

· But it is not always the case that if $f''(z) = 0$, then z is an inflection point. For example,

$f(x) = x^4$



$\uparrow$

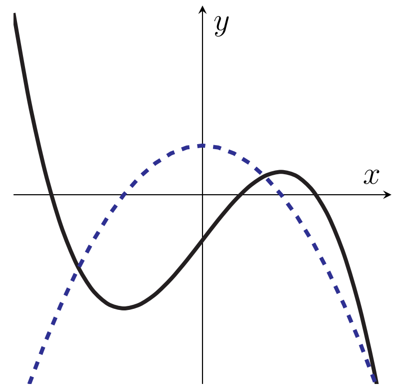
$f'' = 0$

In this example we have a function that is concave up everywhere, and at $x = 0$ we have $f''(0) = 0$. So in this case we don't have an inflection point at $x = 0$, since there is no change in concavity.

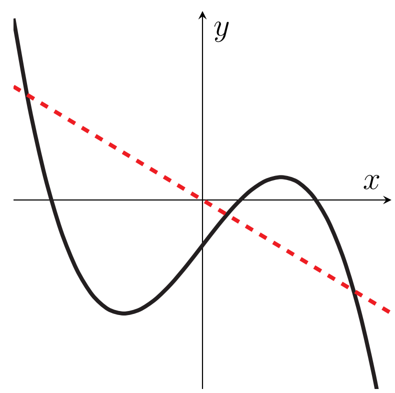
This should also serve as a warning, that just because we know a graph is concave up everywhere, doesnt mean we know that f'' is strictly greater than 0, since we have just seen that f'' can be equal to 0. This gives rise to the following rule:

· If the graph of f is concave up on an interval, then $f'' \geq 0$ on that interval.
· If the graph of f is concave down on an interval, then $f'' \leq 0$ on that interval.

Example 3: For the following graph of a function f, graph its first derivative f' and its second derivative f'' and relate the resulting graphs.



Graph of function f in black, f' in blue.



Graph of function f in black, f'' in red.

As you can see, when the graph of f'' is above the x-axis, the graph of f is concave up. And when the graph of f'' is below the x-axis, the graph of f is concave down. At $x = 0$, the graph of f changes from concave up to concave down. So we have an inflection point at $x = 0$, and as you can see we have $f''(0) = 0$.

Velocity and Acceleration:

In Chapter 2.1 we discussed that given some object that moves according to some position function over time $s(t)$, the velocity of that object at some point $x = a$, would be the instantaneous rate of change (or the derivative) at that point, which we characterized by $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. So the velocity function $v(t)$ of this object's movement over time would be given by the derivative of $s(t)$.

But we are often in situations where velocity also has a rate of change, for example speeding up and slowing down in a car. We call this rate of change *acceleration*. And for any velocity function over time $v(t)$ we have:

$$\text{Instantaneous acceleration} = v'(t) = \lim_{h\to 0} \frac{v(t+h)-v(t)}{h}$$

So the acceleration function $a(t)$ of an object's movement over time would be given by the derivative of $v(t)$. But remember that $v(t)$ is the derivative of $s(t)$. So our acceleration function $a(t)$ would be the second derivative of our position function $s(t)$. Summarizing:

$$\text{Position as a function of time: } s(t)$$
$$\text{Velocity: } v(t) = s'(t)$$
$$\text{Acceleration: } a(t) = v'(t) = s''(t)$$

Example 4: If a rock is thrown upward on Earth, with an initial velocity of 32 ft/sec from the top of a 48 ft building, we can model the height of the rock at time t by $s(t) = -16t^2 + 32t + 48$. After throwing the rock up, it will at some point be level again with the top of the 48 ft building.

i) What will the velocity be at this time?

ii) What will the acceleration be?

Solution:
i) Before we start dealing with velocity, we should first find the point t_0 where the rock will be level again with the top of the bulding. We know that at this point t_0, my position function will be equal to 48 (the height of my building), so set

$$48 = -16t^2 + 32t + 48$$
$$\Rightarrow 0 = -16t^2 + 32t$$
$$\Rightarrow 0 = t(-16t + 32)$$

So at $t = 0$ and at $t = 2$ seconds, we have the rock level with the building. But note that $t = 0$ is where we start, and the question asks when will the rock be level with the building *again*. So we know that $t_0 = 2$ seconds.

Now that we have the point t_0, we want to find the velocity equation $v(t)$, and simply plug in $t_0 = 2$. To do this we must remember that the velocity function $v(t)$ is equal to the derivative function $s'(t)$. So,

$$v(t) = s'(t) = \lim_{h\to 0} \tfrac{s(t+h)-s(t)}{h}$$

$$\Rightarrow v(t) = \lim_{h\to 0} \tfrac{[-16(t+h)^2+32(t+h)+48]-[-16t^2+32t+48]}{h}$$

$$\Rightarrow v(t) = \lim_{h\to 0} \tfrac{[-16t^2-32th-16h^2+32t+32h+48]-[-16t^2+32t+48]}{h}$$

$$\Rightarrow v(t) = \lim_{h\to 0} \tfrac{-32th-16h^2+32h}{h}$$

$$\Rightarrow v(t) = \lim_{h\to 0}[-32t - 16h + 32]$$
$$\Rightarrow v(t) = -32t + 32$$

$$\text{So } v(2) = s'(2) = -64 + 32 = -32$$

Therefore the veloctiy at this point will be -32 ft/sec (note that this number being negative, represents the fact that the rock is on its way down, instead of on its way up).

ii) Now we want to find the acceleration function $a(t)$. But remember the acceleration function $a(t)$ is equal to the derivative of the velocity function $v(t)$. So,

$$a(t) = v'(t) = \tfrac{d}{dt}[-32t + 32] = -32$$
$$(\text{since } v(t) \text{ is a line with slope } -32)$$

So acceleration is actually constant, and is always equal to -32 ft/sec. Therefore at $t_0 = 2$, the acceleration will be -32 ft/sec.