MATH 1300 Lecture Notes for Section 2.4 Friday, September 20, 2013

From the previous section, we learned that $f'(x)$ is the derivative function, which gives the slope of the tangent line to the function at any given x-value. The purpose of this section is to show another notation for the derivative, and to practice interpreting the meaning of the derivative in real examples.

Look at the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Notice that if $y = f(x)$, then the numerator is the change in the y value of the function, which we can call Δy, and the denominator, h, is the change in the x value, which we can call Δx.

$$f'(x) = \lim_{h\to 0} \frac{\text{change in y}}{\text{change in x}} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$$

This inspires a new way to write a derivative: $\frac{dy}{dx}$ means the derivative of y as a function of x. Since $y = f(x)$, we also write $\frac{df}{dx}$. This new notation is very useful for figuring out the units for the derivative. And the units can help us interpret the meaning.

Writng $\frac{dy}{dx}$ has one disadvantage: it doesn't clearly show that the derivative is a function. To get around this, for $f'(2)$ we write $\frac{dy}{dx}|_{x=2}$, meaning that we substitute $x = 2$ into the derivative function.

Writing $\frac{dy}{dx}$ for the derivative emphasizes that it is a slope, because it looks like "change in y over change in x". The units of "change in y" are the same as the units of y (the output of the function) and the units of "change in x" are the same as the units of x (the input to the function). Knowing what the units are for $f'(x)$ can really help you interpret what it means.

Example: Say $f(t)$ represents the price in dollars of an ounce of gold in year t. What do $f(2010)$ and $f'(2010)$ represent, including units?
Answer: First notice that price is a function of time, so $P = f(t)$, and $f'(t) = \frac{dP}{dt}$.
$f(2010)$ is the price, in dollars, of an ounce of gold in year 2010.
The units of $f'(x) = \frac{dP}{dt}$ are dollars/year. f'(2010) is the rate of change of the price of gold in 2010, in dollars per year. It's how fast the price of gold is changing in the year 2010.

Example: The function $s(t)$ gives the inches of snow that have fallen in the t hours since midnight, during one of Boulder's epic snowstorms. Determine the meaning of each of these statements:

1. $s(1) = 2.3$
   The input is $t = 1$ hour, the output is s = 2.3 inches. So at 1:00 am it has snowed 2.3 inches.

2. $s'(2) = 1.8$
   $s'(t) = \frac{ds}{dt}$, so the units are inches per hour, and it measures the rate that the snow is falling. At $t = 2$ (2:00 am) the snow is falling at a rate of 1.8 inches per hour. If it were to continue falling at exactly that rate, then we would see an additional 1.8 inches in the next hour. (But of course 1.8 inches per hour is only the rate at that instant, and we fully expect it to vary over the next hour.)

3. $s^{-1}(5) = 4.5$
   First notice that s measures how much snow has fallen, so it is an increasing function, passes the horizontal line test, and is invertible. The input is snowfall (in inches) and the output is time (in hours). So it takes 4.5 hours for the first 5 inches of snow to fall.

4. $(s^{-1})'(7) = .75$
   $(s^{-1})' = \frac{dt}{ds}$, the units are hours per inch. When 7 inches have fallen, the rate is such that it takes .75 hours for one inch of snow to fall.