1. Section 2.3 of HH - The Derivative Function

   The major difference between this section 2.3 (The Derivative Function) and the previous section 2.2 (The Derivative at a Point) is that instead of computing the slope of the tangent line to the graph of f at one point on the graph of f, and then perhaps at other points on the graph of f, each time requiring the evaluation of the limit of the difference quotient for f, we now derive a *function* called f' which allows us to compute the slope of the tangent line to the graph of f at any point on the graph of f simply by evaluating the function f' at the x-coordinate of the point of tangency.

   In section 2.2, given a particular numerical value of a, you computed the derivative of f at a, denoted by $f'(a)$, using the formula:

   $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

   where a is a number in the domain of f, and the result $f'(a)$ is a number which is the slope of the tangent line to the graph of f at the point $(a, f(a))$.

   In section 2.3, simply substitute a letter like x for the number a and compute:

   $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

   where x is a variable (a letter that stands for a number), and the result $f'(x)$ is an expression involving x, a *function* of x called *the derivative of f at x*, or simply *the derivative of f*. To find the slope of the tangent line to the graph of f at the point on the graph $(b, f(b))$, simply evaluate the function $f'(x)$ letting $x = b$; i.e. evaluate $f'(b)$. Computing the function f' gives you a formula for the slope of the tangent line anywhere on the graph of f, and you only have to evaluate the limit of the difference quotient for f one time in order to find the derivative function f'.

   Interpreting $f'(x)$ as the slope of the tangent line to the graph of f at the point whose first coordinate is x is only one way to interpret what the derivative function f' is telling you. More generally, $f'(x)$ is the rate at which the output of f is changing at the point on the graph of f where the first coordinate is x. This is the meaning of "Rate of change of f at x" in the blue box on page 86 in your textbook.

   Use the following image to help you understand the connection between the slope of a tangent line to the graph of f and the rate of change of the output of f. Where the graph of f is fairly flat, the slope of a tangent line will be a fairly small number, and the rate at which the output of f is changing will be fairly small as x moves along the input axis. But where the graph of f is rising sharply, the slope of a tangent line will be a large number, and the rate at which the output of f is changing will be big as x moves along the input axis. The slope of the tangent line to the graph of f at the point $(x, f(x))$ *is numerically equal to* the rate at which the output of the function f is changing at the point $(x, f(x))$.

Given a function f, the derivative function f' is explored in Section 2.3 in three ways: graphically, numerically, and from a formula.

(a) Graphically
Since f' is a function, f' has a graph. As you might expect, the graphs of f and f' are closely related to one another, as it should be since f' is derived from f (this is why the function f' is called the derivative of f).

Look carefully at Figure 2.30 on page 86 of of your textbook. The blue graph is the graph of f, and the black graph is the graph of f'. For any given x, the slope of the tangent line to the graph of f is the value of $f'(x)$, the distance up or down to the graph of f' above or below the given x.

For example, $f'(4) = 1$ in Figure 2.30. This means that the slope of the tangent line to the graph of f at the point $(4, f(4))$ is equal to $f'(4) = 1$.

As another example, if you want to know the slope of the tangent line to the graph of f at the point $(0, 4)$, that slope will equal $f'(0) = -1$.

Now notice the intervals along the x-axis above or below which the graph of f is increasing (which means going up and to the right): these are the intervals $(-2, -\frac{1}{2})$ and $(\frac{7}{2}, 5)$. Notice that these are exactly the intervals for which $f'(x) > 0$.

And notice the interval along the x-axis above or below which the graph of f is decreasing (which means going down and to the right): this is the interval $(-\frac{1}{2}, \frac{7}{2})$ Notice that this is exactly the interval for which $f'(x) < 0$.

In summary, if $f' > 0$ for every x in an interval, then f is increasing for every x in that interval. And if $f' < 0$ for every x in an interval, then f is decreasing for every x in that interval.

(b) Numerically
Sometimes, the values of a function f are given explicitly in a table, rather than in a graph or with a formula. The advantage of a table of values is that you can just read from the table what the output values of the function are for the given inputs; the disadvantage of a table of values is that you do not know what the output of the function is for values of the input that are not given in the table.

Look carefully at the table of values in Example 3, Table 2.7 on page 87 of your textbook. Here you see the specification of a function called c, with input values in units of minutes specified by the independent variable t, and the outputs giving the concentration $c(t)$ of a drug in the bloodstream at time t. Notice that the values of t are given from $t = 0$ minutes to $t = 1$ minutes, in increments of 0.1 minutes. You can read from the table that $c(0.3) = 0.98$ for example, but there is no explicit information given in the table about $c(0.35)$. How do we compute values for the derivative $c'(t)$ for various t's?

The method for computing $c'(t)$ for the values of t specified in the table is demonstrated in the lower quarter of page 87 in your textbook. The method is based on the definition of the derivative of the function c, using t (rather than x) as the independent variable:

$$c'(t) = \lim_{h \to 0} \frac{c(t+h) - c(t)}{h}$$

This means that the closer h gets to 0, the closer the difference quotient $\dfrac{c(t+h) - c(t)}{h}$ gets to $c'(t)$. Symbolically,

$$c'(t) \approx \frac{c(t+h) - c(t)}{h} \text{ when } h \approx 0.$$

But because the values of t in the table are spaced 0.1 minutes apart, it's not possible to compute $c(t+h)$ for values of h arbitrary close to 0. For example, if $t = 0.3$ and we were trying to compute $c'(0.3)$, we couldn't compute $c(t+h) = c(0.3 + 0.001) = c(3.001)$, where $h = 0.001$, because there is no value in the table given for $c(3.001)$. For any value of t given in the table, the smallest positive value for h we could use would be $h = 0.1$. This is the best we can do when trying to compute $c'(t)$ for the values of t given in the table. And that's exactly what you see near the bottom of page 87. Study carefully these numerical approximations for $c'(t)$ for the values of $t = 0, 0.1, 0.2, 0.3$, and 0.4. The complete set of approximate values of $c'(t)$ is given at the top of page 88, for t from 0 to 0.9 in steps of 0.1.

Notice the graph of the $c(t)$ in Figure 2.31, with the values for $c(t)$ from the Table 2.7 shown as the black dots in Figure 2.31, and the "in-between" points on the graph of $c(t)$ filled in as a good guess as to what a complete graph for $c(t)$ probably looks like, connecting the dots by sections of smooth curve. Notice in particular how the slopes of the tangent lines to the graph of $c(t)$ in Figure 2.31 are shown in the Table 2.8 as the values of the derivative $c'(t)$.

At the right end of Table 2.7, for $t = 1.0$, the approximation of $c'(1)$ as:

$$c'(1) \approx \frac{c(1+h) - c(1)}{h} \text{ for } h = 0.1$$

is problematic since there is no value of $c(1 + 0.1) = c(1.1)$ in the table of values for $c(t)$. But there is no reason we couldn't use $h = -0.1$, so that:

$$c'(1) \approx \frac{c(1+h) - c(1)}{h} = \frac{c(1-0.1) - c(1)}{-0.1} = \frac{c(0.9) - c(1)}{-0.1}$$

$$= \frac{0.63 - 0.41}{-0.1} \approx -3.5$$

which would make a reasonable final entry in Table 2.8.

There is a way to improve the approximation of $c'(t)$ for values of t in the middle of the table by averaging the slopes to the left and to the right of a given t. So having done the calculations like those at the bottom of page 87, the results of which are shown in Table 2.8 at the top of page 88, use those slopes in Table 2.8 to perform averaging like that shown in the middle of page 88 to approximate $c'(0.2)$. The resulting value of $c'(0.2) \approx 0.45$ shown in the middle of page 88 is a better approximation to $c'(0.2)$ than the $c'(2) \approx 0.4$ shown in Table 2.8.

(c) Formulas
Given a formula for the function f, the only way you now know to compute the derivative function f' is from the definition of the derivate of f, that is by taking the limit of the difference quotient for f:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Carrying out this process, which is known as 'differentiation', involves increasing amounts of algebra (and later, trig) as the function f gets more and more complicated. Much of Chapter 3 will be devoted to developing short-cuts for computing the derivative function f' from f, the short-cuts being based on the above definition of the derivative but being much simpler to carry out, involving much less algebra.

The examples below demonstrate the process of differentiation for some relatively simple functions, all using the definition of the derivative..

2. Examples

(a) The constant function

The constant function is perhaps the simplest of all functions, the function which outputs the constant real number k, completely ignoring the input x:

$$y = f(x) = k$$

The constant function is referred to as <u>the</u> constant function, but actually there are infinitely many constant functions, one for each choice of the real number k. So $f(x) = 3$ and $f(x) = -17$ are two examples of specific constant functions. But it is useful to group all of the infinitely many constant functions into a single generic constant function $f(x) = k$ and call it the constant function.

The graph of the constant function has y-coordinate equal to k, no matter what the value of x is. This in turn means that the graph of the constant function $y = f(x) = k$ is a horizontal line that passes thru the point $(0, k)$ on the y-axis.

If you are given some line, then the only sensible meaning for the tangent line to that given line is the line itself. In the case of the constant function, the given line is the horizontal line that passes thru $(0, k)$, so the tangent line to the graph of the constant function (a horizontal line) is the horizontal line itself. Since this line is horizontal for all x and so has slope equal to 0 for all x, it shouldn't come as too much of a surprise that the derivative function for the constant function $f(x) = k$ is:

$$f'(x) = 0 \text{ for all } x$$

Let's see that this is indeed the case, using the definition of the derivative, and keeping in mind that the output of the constant function $f(x) = k$ is always the real number k, no matter what the input is.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{k-k}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0}(0) = 0$$

In words, the derivative of the constant function is the constant function that always outputs zero, no matter what the input is. Zero is the slope of the tangent line anywhere along the graph of the constant function $f(x) = k$, which is equivalent to saying that the rate at which the output of the constant function changes is zero, because the output of the constant function never changes; the output is always k. Again, in words, the derivative of the constant function is the constant function which always outputs zero. In a fewer number of words (but not as precisely), we say that the derivative of the constant function is zero. In even fewer words (and with even less precision), we say that the derivative of any constant is zero.

(b) The linear function

The linear function is given by $f(x) = mx + b$, whose graph is a line with slope m and y-intercept b. This is an infinite family of functions, one for each choice of the constants m and b. An example of a specific linear function is $f(x) = 3x + 2$. But like the constant function, it is useful to group all of the infinitely many linear functions into a single generic linear function $f(x) = mx + b$ and call it the linear function.

Since the graph of the linear function $f(x) = mx + b$ is a line with slope m, and since the tangent line to a line is the line itself, it seems reasonable to guess that the slope of the tangent line to the graph of the linear function $f(x) = mx + b$ should be the constant m, no matter what the value of x is. In other words, we should expect that for the linear function $f(x) = mx + b$, the derivative function should be $f'(x) = m$, for all x, which is the constant function that outputs the constant m no matter what the input x is. This is borne out by the following differentiation calculation using the limit of a difference quotient for the linear function $f(x) = mx + b$:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} && \text{definition of derivative of } f \\
&= \lim_{h \to 0} \frac{[m(x+h) + b] - [mx + b]}{h} && \text{since } f(x+h) = m(x+h) + b \\
&= \lim_{h \to 0} \frac{mx + mh + b - mx - b}{h} && \text{multiply everything out} \\
&= \lim_{h \to 0} \frac{mh}{h} && \text{the terms } mx \text{ and } b \text{ subtract away} \\
&= \lim_{h \to 0} (m) && \text{divide numerator and denominator by (the non-zero) } h \\
&= m && \text{since the limit of any constant is the constant}
\end{aligned}$$

(c) The squaring function

The squaring function is given by $f(x) = x^2$
It turns out that the derivative function for the squaring function is $f'(x) = 2x$. You can find the details of this differentiation calculation, using the definition of the derivative as the limit of a difference quotient, in the last 6 lines of Example 4 on page 89 in your textbook. You should study this calculation carefully and be sure you understand all the details.

(d) The cubing function

The squaring function is given by $f(x) = x^3$
It turns out that the derivative function for the cubing function is $f'(x) = 3x^2$. You can find the details of this differentiation calculation, using the definition of the derivative as the limit of a difference quotient, in Example 5 starting near the bottom of page 89 in your textbook and continuing on to the top of page 90. You should study this calculation very carefully and be sure you understand all the details.

(e) The nth-power function

The nth-power function is given by $f(x) = x^n$, where n is any real constant. The previous two examples are the special cases where $n = 2$ (the squaring function) and $n = 3$ (the cubing function). You will discover that the derivative of the nth-power function is $f'(x) = nx^{n-1}$. This is called the power rule and is one of the most wonderful shortcuts that will be developed in Chapter 3.

Notice that for $n = 2$ (the squaring function), $f'(x) = nx^{n-1} = 2x^{2-1} = 2x$ in accordance with the result for the squaring function above.

And for $n = 3$ (the cubing function), $f'(x) = nx^{n-1} = 3x^{3-1} = 3x^2$ in accordance with the result for the cubing function above.

(f) An arbitrary quadratic function
For the quadratic function $f(x) = 3x^2 - 5x + 6$, here are the details of the calculation of the derivative function $f'(x)$. The calculation shows that $f'(x) = 6x - 5$. Be sure you understand the reason for each step in the calculation.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[3(x+h)^2 - 5(x+h) + 6] - [3x^2 - 5x + 6]}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 5x - 5h + 6 - 3x^2 + 5x - 6}{h} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2 - 5h}{h} \\
&= \lim_{h \to 0} \frac{h(6x + 3h - 5)}{h} \\
&= \lim_{h \to 0} (6x + 3h - 5) \\
&= 6x - 5
\end{aligned}$$