

MATH 1300

Lecture Notes Wednesday, Sept. 11, 2013.

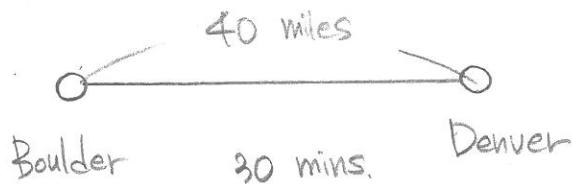
## Section 2.1. How do we measure speed?

Measuring a speed is a classical question in the history of mathematics and physics. More precisely, people had the following question:

how do we measure the speed of an object at an exact time?

Let's consider the following example.

ex) You are driving from Boulder to Denver to watch the Rockies game. The distance between two cities is approximately 40 miles. and it took you 30 minutes to drive there. What was your speed at 10 minutes after you left Boulder?

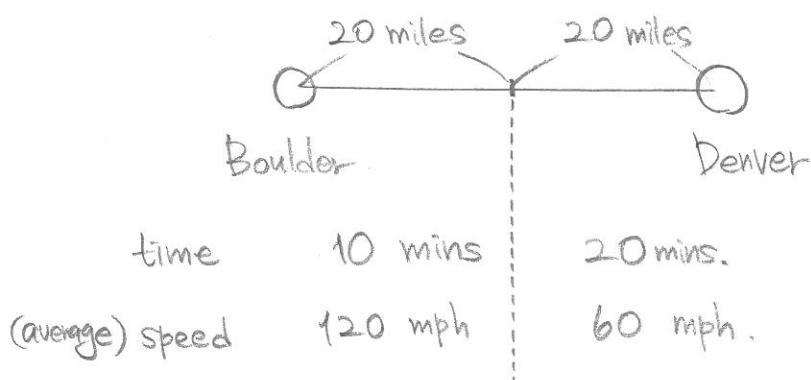


You might recall the speed formula

$$\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{40}{0.5} = 80 \text{ mph}$$

and conclude that the speed at 10 minutes was 80 mph. But this is actually an 'average speed'. So your speculation may be far from the truth.

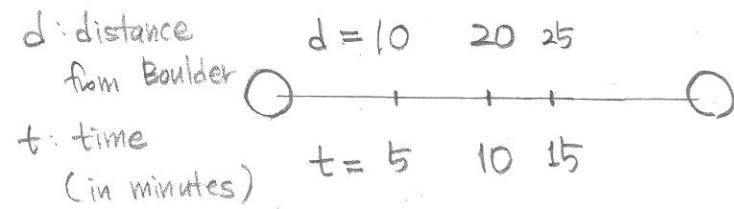
Consider the following case.



You drive the first half in 10 minutes and the second half in 20 minutes. Then the (average) speeds at each interval is 120 mph, 60 mph, respectively.

So it's hard to believe that your exact speed at 10 minutes was 80 mph. Then how can we measure this speed?

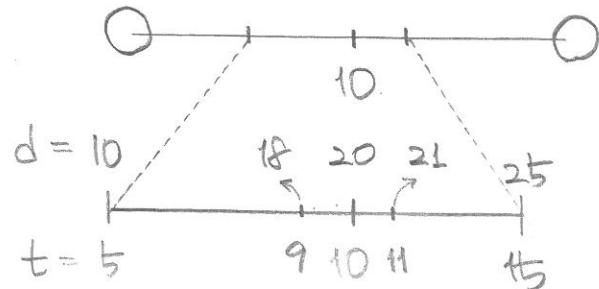
From the above example, we learned that we can figure out the average speed even if we can't measure the instantaneous speed at an exact time. So let's use average speed to estimate the instantaneous speed at  $t=10$ . A reasonable approach would be to look at the average rate of change around  $t=10$ .



From the data on the interval of time from  $t=5$  to  $t=15$ , we find the average speed on that interval is

$$s = \frac{15}{0.16} = 93.75 \text{ mph.}$$

By zooming in to a smaller interval around  $t=10$ , we can improve our estimation.



From the data on the interval from  $t=9$  to  $t=11$ , the average speed  $s$  is

$$s = \frac{3}{0.033} = 90.9 \text{ mph}$$

We can continue this process to get a finer estimation, but now we can say that the exact speed at  $t=10$  is around 90 mph with more confidence.

Now, to describe more general motion of an object, we use the notion of velocity rather than speed. Velocity is the rate of change of the "position" of an object and it describes how fast and in what direction an object moves. In fact, speed is the magnitude or absolute value of the velocity.

We define the average velocity as following.

Def average velocity.

$s(t)$  : position of an object at time  $t$ .

the average velocity of an object from  $t=a$  to  $t=b$  is

$$\frac{\text{change in position}}{\text{change in time}} = \frac{s(b) - s(a)}{b - a}.$$

Alternatively, we can denote  $b=ath$ , where  $h$  is the time difference, and write

$$\frac{s(b) - s(a)}{b - a} = \frac{s(ath) - s(a)}{(ath) - a} = \frac{s(ath) - s(a)}{h}.$$

Just as we used average speed to estimate the instantaneous speed in the previous example, we take average velocity to measure the instantaneous velocity. As we observed, we can improve the estimation by taking shorter intervals of time. To achieve the most accuracy, we use the notion of limit to describe the instantaneous velocity.

Def instantaneous velocity.

$s(t)$  : position of an object at time  $t$ .

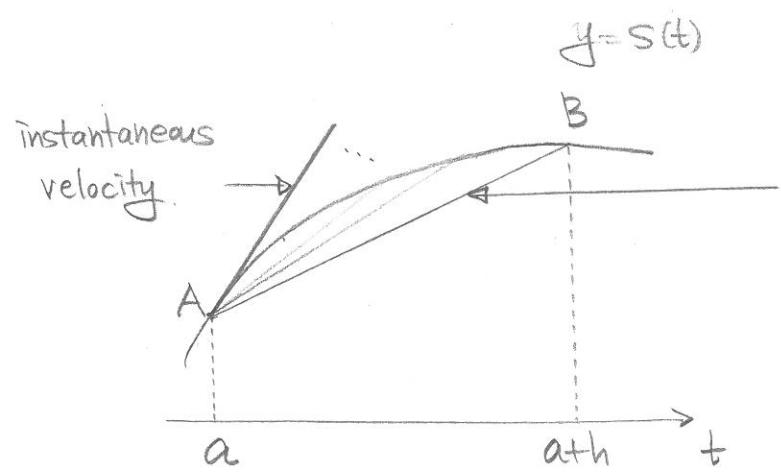
the instantaneous velocity of an object at  $t=a$  is the limit

$$\lim_{h \rightarrow 0} \frac{s(ath) - s(a)}{h}.$$

ex) Suppose that the position of an object is given by the function  $s(t) = \frac{1}{t+1}$ . Then the instantaneous rate of change at  $t=2$  is

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{s(2+h) - s(2)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{3 - (3+h)}{3(3+h)}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{3(3+h)} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{3(3+h)} \\
 &= -\frac{1}{9}.
 \end{aligned}$$

Consider the graph of the function  $y = s(t)$ , where  $s(t)$  is the position of an object at time  $t$ .



We can see that the average velocity from  $t=a$  to  $t=ath$  is the slope of the secant line passing through points A and B.

As  $h \rightarrow 0$ , we can also see that instantaneous velocity approaches to the slope of tangent line at  $t=a$ .

ex) Suppose that the position of an object is given by the function  $s(t) = \sqrt{t}$ . Find the slope of the tangent line at  $t=3$ .

sol

The slope of the tangent line at  $t=3$  is the instantaneous velocity at  $t=3$ , which is

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{s(3+h) - s(3)}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{3+h} - \sqrt{3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{3+h} - \sqrt{3}}{h} \cdot \frac{\sqrt{3+h} + \sqrt{3}}{\sqrt{3+h} + \sqrt{3}} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{3+h})^2 - (\sqrt{3})^2}{h(\sqrt{3+h} + \sqrt{3})} \\ &= \lim_{h \rightarrow 0} \frac{(3+h) - 3}{h(\sqrt{3+h} + \sqrt{3})} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{3+h} + \sqrt{3})} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{3+h} + \sqrt{3}} \\ &= \frac{1}{\sqrt{3+0} + \sqrt{3}} = \frac{1}{2\sqrt{3}}. \end{aligned}$$

## Section 2.2. The derivative at a point.

In the previous section, we discussed the average velocity and instantaneous velocity of an object whose position is given by the function  $y = s(t)$ . In this section, we generalize these notions into the average rate of change and instantaneous rate of change of an arbitrary function  $y = f(x)$ .

### Def average rate of change

For the function  $y = f(x)$ , the average rate of change of  $f$  over the interval from  $a$  to  $b$  is

$$\frac{\text{change of } y}{\text{change of } x} = \frac{f(b) - f(a)}{b - a}.$$

Alternatively, we can denote  $b = a + h$ , where  $h$  is the length of interval  $[a, b]$ , and write

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

Def instantaneous rate of change / derivative.

For the function  $y=f(x)$ , the instantaneous rate of change of  $f$  at  $x=a$  is defined as

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

We call it the derivative of  $f$  at  $a$  and denote it by  $f'(a)$ .

ex) Find the derivative of the function  $f(x)=x^2-x$  at  $x=-2$ .

sol

$$\begin{aligned} f'(-2) &= \lim_{h \rightarrow 0} \frac{f(-2+h) - f(-2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{((-2+h)^2 - (-2+h)) - ((-2)^2 - (-2))}{h} \\ &= \lim_{h \rightarrow 0} \frac{(-4h + h^2 + 4 - h) - 4}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 - 5h}{h} \\ &= \lim_{h \rightarrow 0} (h - 5) \\ &= -5. \end{aligned}$$

Just as in the case of average/instantaneous velocity, average/instantaneous rate of change of the function  $f$  corresponds to the slope of secant/tangent line of  $f$ .

ex) Find the equation of tangent line for the function

$$g(x) = \frac{1}{x^2} \text{ at } x=5.$$

Sol

To find the equation of the tangent line, we need to find the slope and the point that the line is passing through.  $g(5) = \frac{1}{25}$ , so the tangent line passes through the point  $(5, \frac{1}{25})$ . The slope of the tangent line at  $x=5$  is

$$\begin{aligned} g'(5) &= \lim_{h \rightarrow 0} \frac{g(5+h) - g(5)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{(5+h)^2} - \frac{1}{5^2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{25 - (25 - 10h + h^2)}{25(25 + 10h + h^2)} \\ &= \lim_{h \rightarrow 0} \frac{10h + h^2}{25(25 + 10h + h^2)} \\ &= \lim_{h \rightarrow 0} \frac{10 + h}{25(25 + 10h + h^2)} = \frac{10}{25 \cdot 25} = \frac{2}{125}. \end{aligned}$$

Hence, the equation of the tangent line is

$$y = \frac{2}{125}(x - 5) + \frac{1}{25}$$