Math 1300: Calculus 1
Section 1.8: Limits
Sep 9-10, 2013

Limits

We say the limit of $f(x)$ as x approaches c is L or

$$\lim_{x \to c} f(x) = L$$

if $f(x)$ can be made as close as we like to L by choosing x sufficiently close to (but not equal to) c.

Note: $f(c)$ itself need not equal L. If $f(x) = L$ then f is continuous at c.

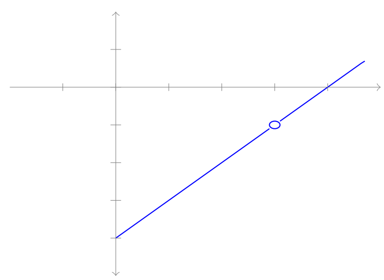
Example 1. *Let* $f(x) = \frac{x^2-7x+12}{x-3}$. *We can evaluate*

$$\lim_{x \to 3} f(x)$$

by factoring the numerator. Notice that we get

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - 7x + 12}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x-4)}{(x-3)} = \lim_{x \to 3} x - 4 = -1.$$

We can look at the graph and see that even though the function is not defined at $x = 3$, *and thus not continuous, the limit still exist.*



Horizontal Asymptote

The **horizontal asymptote** is the limit as $x \to \pm\infty$.

Example 2. *We can find the horizontal asymptote of the rational function*

$$f(x) = \frac{3x^2 - 2}{x^2 + 1}$$

by evaluating the limit as $x \to \infty$. *We learned in Section 1.6 that if the degree of the numerator is higher than the degree of the denominator then there will be no horizontal asymptote, and if the degree of the numerator is lower than the degree of the denominator, then the horizontal asymptote will be* $y = 0$, *and if the degree of the numerator is the same as the degree of the denominator, then the horizontal asymptote will be the* $y =$ *the ratio of the leading coefficients. We can show this algebraically now using limits. Multiply numerator and denominator by* $\frac{1}{\text{highest power in denominator}}$.

$$\lim_{x \to \infty} \frac{3x^2 - 2}{x^2 + 1} =$$

$$\lim_{x\to\infty} \frac{(3x^2-2)\frac{1}{x^2}}{(x^2+1)\frac{1}{x^2}} =$$

$$\lim_{x\to\infty} \frac{3-\frac{2}{x^2}}{1+\frac{1}{x^2}} = \frac{3-0}{1-0} = 3$$

Thus $f(x)$ has a horizontal asymptote of $y = 3$.

Properties of Limits

- If b is a constant, then $\lim_{x\to c}(bf(x)) = b\left(\lim_{c\to c} f(x)\right)$
- $\lim_{x\to c} f(x) + g(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
- $\lim_{x\to c} f(x)g(x) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$
- $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim g(x) \neq 0$.
- $\lim_{x\to c} k = k$ for any constant k.

Example 3. *Notice that to be able to use the division property of the limit we must require that the limit of the denominator not be equal to zero. For example,*

$$\lim_{x\to 2} \frac{x-2}{x^2-4}$$

we see that $\lim_{x\to 2} x^2-4 = 0$ *and thus we cannot apply the property of the limit. First we must simplify the expression,*

$$\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{1}{x+2} = \frac{\lim_{x\to 2} 1}{\lim_{x\to 2} x+2} = \frac{1}{4}.$$

Example 4. *In this case can apply the properties of a limit directly*

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2+5x}{x+9} &= \frac{\lim_{x\to 3}(x^2+5x)}{\lim_{x\to 3}(x+9)} \\
&= \frac{\lim_{x\to 3} x^2 + 5\cdot\lim_{x\to 3} x}{\lim_{x\to 3} x + \lim_{x\to 3} 9} \\
&= 2
\end{aligned}$$

Proposition 1. *We have the useful trigonometric limit*

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1.$$

The basic algorithm for solving limits is:

1. Evaluate the function if possible. In other words, just substitute to get the limit.
2. If there is a zero in the denominator and the numerator value is not zero we have a vertical asymptote. In which case the limit must be taken from left and right separately, each one will be either $+\infty$ or $-\infty$.
3. If there is a zero in both the numerator and denominator we have to do some more work. We might use a factor/cancel simplification, or common denominators if we are dealing with fractions inside of fractions, or conjugation if we are dealing with radicals.
4. Trig limits might require simplification, then using the limit given above.

One-sided Limits

Typically when we write

$$\lim_{x \to c} f(x)$$

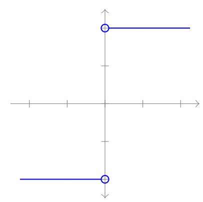
we consider values of $f(x)$ as x approaches c from values greater and less than c. If we only want to consider values of $f(x)$ as x approaches c from values greater than c we write,

$$\lim_{x \to c^+} f(x)$$

which we call the *right-hand limit*. Similarly if we want to consider values of $f(x)$ as x approaches c from values less than c we write

$$\lim_{x \to c^-} f(x)$$

which we call the *left-hand limit*. Consider the following situation,



here we have that

$$\lim_{x \to 0^-} f(x) = -2 \quad \text{and} \quad \lim_{x \to 0^+} f(x) = 2.$$

Theorem 1. *We say*

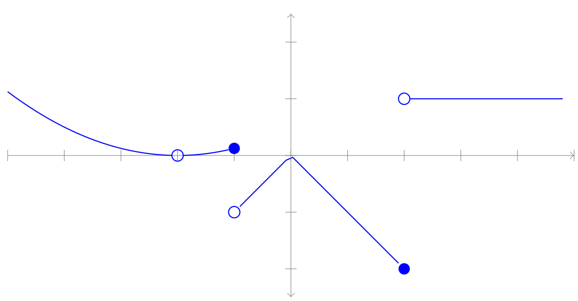
$$\lim_{x \to c} f(x) = L$$

if and only if

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

.

Example 5. *Consider the following function*



at the value $x = -2$ we have that

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x) = 0$$

and so $\lim_{x \to -2} f(x) = 0$. At the value $x = -1$ we have that

$$\lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x)$$

and so $\lim_{x \to -1} f(x)$ does not exist. Similarly

$$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$$

and so $\lim_{x \to 2} f(x)$ does not exist.

1 Continuity

Definition 1. *We say a function f is continuous at $x = c$ if f is defined at $x = c$ and if*

$$\lim_{x \to c} f(x) = f(c).$$

Essentially, this definition implicitly requires three things: the limit must exist, the function must be defined, and those two values must be the same.

Example 6. *Define $f(x)$ as follows,*

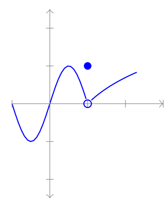
$$f(x) = \begin{cases} \sin(\pi x) & x < 1 \\ 1 & x = 1 \\ \ln(x) & x > 1 \end{cases}$$

. Is $f(x)$ continuous?

Notice that $\sin(\pi x)$ is continuous everywhere, $\ln(x)$ is continuous on its domain and thus if we want to check for continuity we only need to check at the point $x = 1$. Therefore

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sin(\pi x) = 0$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \ln(x) = 0$$

Therefore $\lim_{x \to 1} f(x) = 0$ but notice that $f(1) = 1$ and so $f(x)$ is not continuous at $x = 1$ even though the limit exist.



We know that continuity has a lot to do with limits and therefore we have the following properties which are similar to the properties of limits. Suppose that f and g are continuous functions

- If b is a constant, then $bf(x)$ is continuous.
- $f(x) + g(x)$ is a continuous function.
- $f(x)g(x)$ is continuous.
- $\frac{f(x)}{g(x)}$ is continuous.

Theorem 2. *If f and g are continuous, and if the composite function $f(g(x))$ is defined on an interval, then $f(g(x))$ is continuous on that interval.*