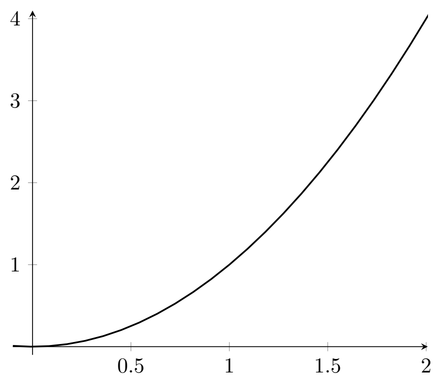
1.7 Introduction to Continuity

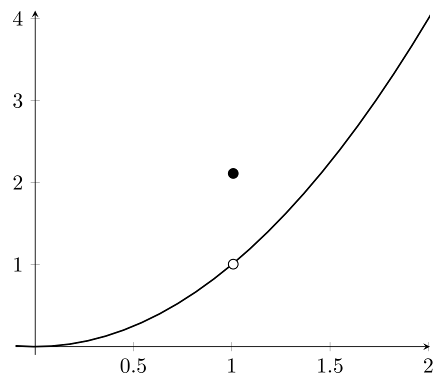
The goal of this section is for us to determine if (and where) a function $f(x)$ is continuous. To begin, here is an informal definition of continuity:

"Definition" 1. A function $f(x)$ is *continuous* if its graph can be drawn without lifting your pencil.

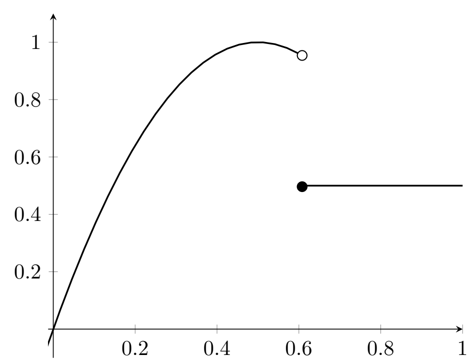
Visually, this means f is continuous if its graph has no jumps, gaps, or holes. A more precise definition will be given once we cover limits, but you should keep this definition in mind for recognizing the graph of a continuous function.

Example 1. The graph on the left is continuous, but the graph on the right is not:





Here is another discontinuous function:



One very important property of continuous functions is described by the following theorem.

Theorem 1 (Intermediate Value Theorem)**.** *If $f(x)$ is a continuous function defined on an interval $[a, b]$, then for any number y in between $f(a)$ and $f(b)$, there is some c so that $f(c) = y$.*

Intuitively, this theorem says that the graph of $f(x)$ can't jump from one y value to another without hitting every y value in between, a property that isn't surprising, given our "definition" of continuity. As an example, if a car drives along a road from town A to town B, then it must drive by every town in between. Just as our hypothetical car cannot teleport past a town in between town A and town B, the graph of a continuous function cannot skip any y values between $f(a)$ and $f(b)$.

Familiar functions from algebra and trigonometry are continuous everywhere on their domains. By familiar functions, we mean the following:

- **Polynomials**, such as $3x^2 - 8x + 2$,
- **Rational functions** (fractions whose numerators and denominators are polynomials), such as $\frac{1}{x}$ and $\frac{x^2+1}{x-7}$,
- **Exponential and logarithmic functions**, i.e., e^x and $\ln(x)$,
- **Trigonometric functions** and **inverse trigonometric functions**, such as $\sin(x)$ and $\tan^{-1}(x)$,
- **Radical functions**, like $\sqrt{3x-9}$ and $\sqrt[3]{x}$, and
- Any function which is composed of any of the above, such as $\frac{e^{\sin(x)}}{\cos(x)}$.

In order to find the domains of these functions, it may be helpful to review Section 1.1, which covers domains in more detail.

Example 2. Let $f(x) = \frac{e^{\sin(x)}}{\cos(x)}$. For which values of x is $f(x)$ continuous?

Solution. As we discussed above, $f(x)$ is continuous for all x in its domain. Thus, we just need to find the domain of $f(x)$. Since exponential and trigonometric functions are defined everywhere, it suffices to find the values of x for which the denominator is zero, i.e., solve the equation

$$\cos(x) = 0.$$

This equation has infinitely many solutions: $\pi/2, 3\pi/2, 5\pi/2$, and in general, $\pi/2 + k\pi$, where k is any whole number. Thus, $f(x)$ is continuous everywhere except for these values of x.

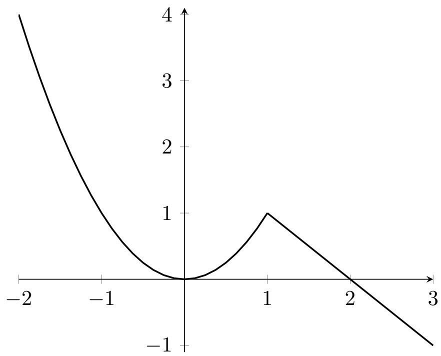
Finally, we talk about *piecewise* functions. These are the functions which have two or more formulas, each defined on some interval. For example,

$$f(x) = \begin{cases} x^2 & : \quad x < 1 \\ 2 - x & : \quad x \geq 1 \end{cases}$$

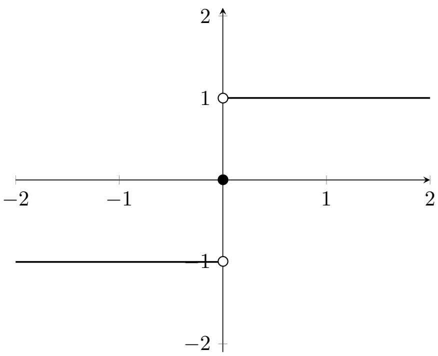
and

$$g(x) = \begin{cases} -1 & : \quad x < 0 \\ 0 & : \quad x = 0 \\ 1 & : \quad x > 0 \end{cases}$$

are piecewise functions. They have the following graphs:



$f(x)$



$g(x)$

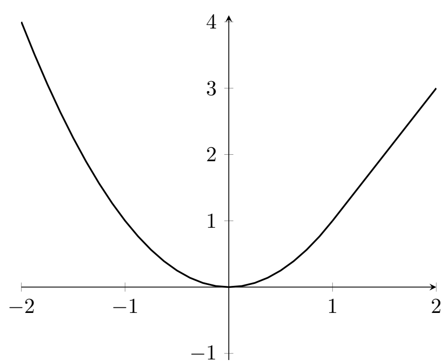
As you can see, the graph on the left is continuous, while the graph on the right is not. In general, piecewise graphs are continuous if the ends of their pieces connect. How do we check if a piecewise function is continuous if we can't look at the graph? The following example provides a method.

Example 3. Let

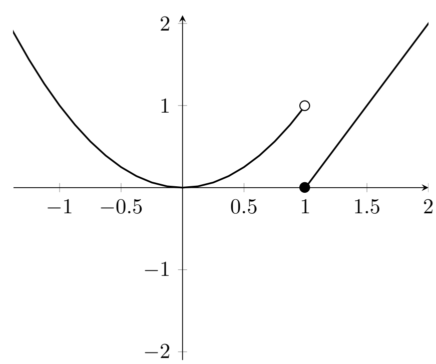
$$f(x) = \begin{cases} x^2 & : \quad x < 1 \\ 2x - k & : \quad x \geq 1 \end{cases},$$

where k is a constant. Is $f(x)$ continuous if $k = 1$? If $k = 2$?

Solution. In both cases, we just need to evaluate each formula at $x = 1$ to check if the graphs meet. At $x = 1$, $x^2 = 1^2 = 1$, so the left "piece" approaches the point $(1, 1)$. At $x = 1$, $2x - k = 2 - k$, so the right "piece" approaches the point $(1, 2 - k)$. Therefore, if we plug in $k = 1$, the resulting graph is continuous, since both pieces approach the same point, $(1, 1)$. Plugging in $k = 2$, the right piece approaches the point $(1, 0)$ while the left piece approaches $(1, 1)$. Thus the graph for $k = 2$ is not continuous. In pictures:



$k = 1$



$k = 2$

In the above solution, we used the notion of a graph "approaching" a certain point. This is the subject of the next section, and we will explore it in much more detail, as well as provide a more precise definition of continuous.