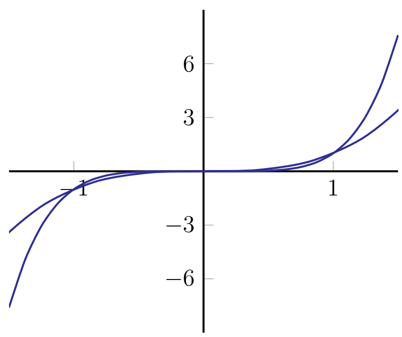
Section 1.6 - Powers, Polynomials, and Rational Functions

Power Functions and Polynomials

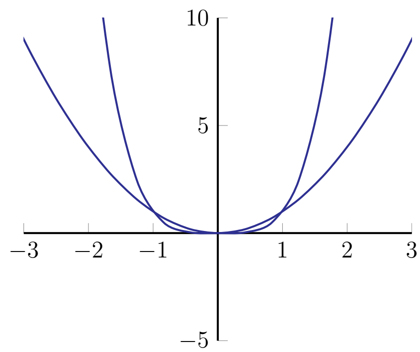
Definition A power function has the form $f(x) = kx^p$ where k and p are constant.

Ex: Volume of a sphere, as a function of radius $V = \frac{4}{3}\pi r^3$

Note: Odd powers are "seat" shaped, even powers are U-shaped.



odd powers of x



even powers of x

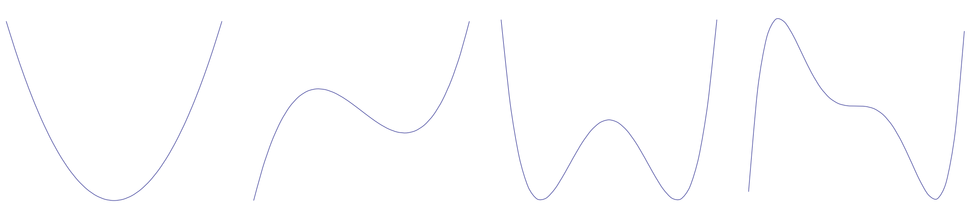
Definition A polynomial is a (finite) sum of power functions with nonnegative integer exponents i.e. it has the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants and n is the degree of the polynomial.

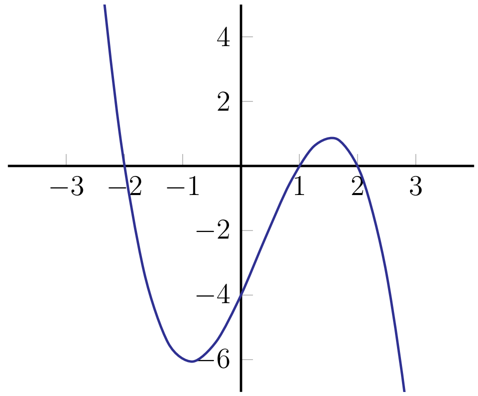
Ex: $f(x) = 8x^{999} + 6x^{100} + 400x^2 + 21x + 5$

Note: An nth degree polynomial " turns around" at most $n-1$ times.



$n = 2$ $\quad$ $n = 3$ $\quad$ $n = 4$ $\quad$ $n = 5$

Ex. Find a possible formula for this polynomial:



1. Cubic polynomial by its shape since the graph crosses the x-axis at each root so there are no double roots

2. Zeros at $x = -2, 1, 2 \implies f(x) = k(x+2)(x-1)(x-2)$

3. Goes through the point $(0, -4)$

$$\begin{aligned} -4 &= k(0+2)(0-1)(0-2) \\ -4 &= 4k \\ k &= -1 \end{aligned}$$

   Because the end behavior the coefficient must be negative.

Therefore, $f(x) = -(x+2)(x-1)(x-2)$.

Rational Functions

Definition A rational function is a ratio of polynomials:

$$f(x) = \frac{p(x)}{q(x)}, \quad p \text{ and } q \text{ polynomials}, \; q(x) \neq 0$$

Definition If the graph of $y = f(x)$ approaches a horizontal line $y = L$ as $x \to \infty$ or as $x \to -\infty$, then the line $y = L$ is called a horizontal asymptote.

Rules for Finding Horizontal Asymptotes

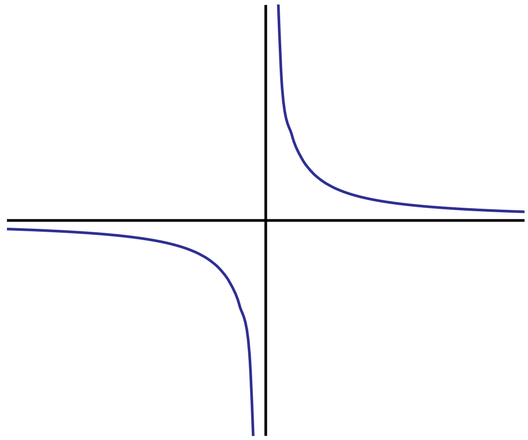
1. If the degree of the numerator is larger than the degree of the denominator, then there is no horizontal asymptote.

2. If the degree of the numerator is smaller than the degree of the denominator then the horizontal asymptote is at $y = 0$.

3. If the degree of the numerator is equal to the degree of the denominator then the horizontal asymptote is the leading coefficient of the numerator divided by the leading coefficient of the denominator.

Definition If the graph of $y = f(x)$ approaches a vertical line $x = K$ as $x \to K$, then the line $x = K$ is called a vertical asymptote.

Ex. $f(x) = 1/x$ has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$.



Ex. Graph $f(x) = \dfrac{2x^2+5}{x^2-25}$

1. The y-intercept: Find $f(0)$

$$f(0) = \frac{2(0)^2+5}{0^2-25} = \frac{-1}{5} \implies \left(0, -\frac{1}{5}\right)$$

2. The x-intercept: Set numerator $= 0$ and solve

$$2x^2 + 5 = 0 \implies \text{no } x\text{-intercepts}$$

3. Vertical asymptotes: Set denominator $= 0$ and solve

$$\begin{array}{rcl} x^2 - 25 & = & 0 \\ (x-5)(x+5) & = & 0 \\ x = 5 & & x = -5 \end{array}$$

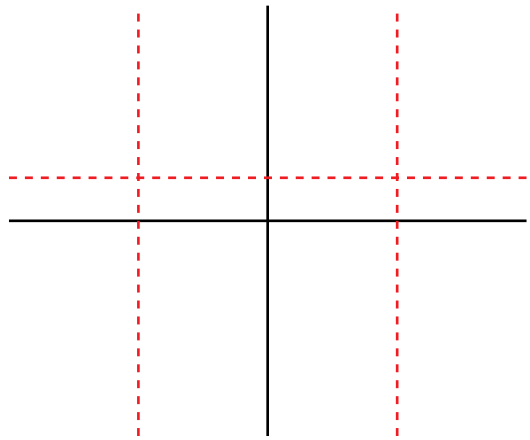
The lines $x = 5$ and $x = -5$

4. Horizontal asymptote: Look at

$$\frac{2x^2}{x^2} = 2$$

The line $y = 2$

Now we're ready to graph... Plotting the asymptotes:



Remember the graph can only cross the y-axis at $(0, -1/5)$ and cannot cross the x-axis at all so we must have: