SECTION 1.5: TRIGONOMETRIC FUNCTIONS

The Unit Circle

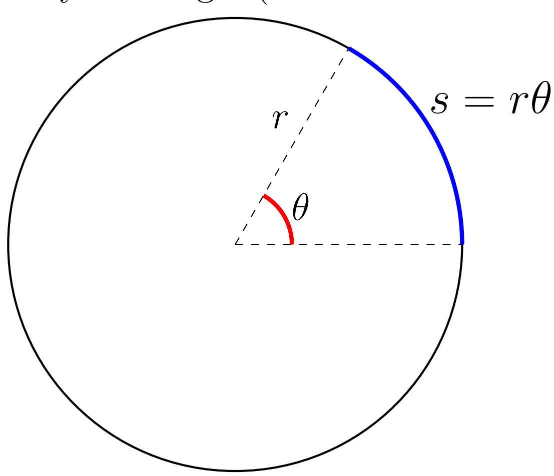
The unit circle is the set of all points in the xy-plane for which $x^2 + y^2 = 1$.

Def: A radian is a unit for measuring angles other than degrees and is measured by the arc length it cuts off from the unit circle. So $360°$ gives you the entire circumference, which is 2π radians. This tells us that in order to convert from degrees to radians, we multiply by a factor of $\frac{2\pi}{360°}$. To convert from radians to degrees we multiply by a factor of $\frac{360°}{2\pi}$.

Ex: $45° = 45° \left(\frac{2\pi}{360°}\right) = 2\pi \left(\frac{45°}{360°}\right) = 2\pi \left(\frac{1}{8}\right) = \frac{\pi}{4}\text{rad}$

$\frac{2\pi}{3}\text{rad} = \frac{2\pi}{3}\left(\frac{360°}{2\pi}\right) = 360° \left(\frac{2\pi}{6\pi}\right) = 360° \left(\frac{1}{3}\right) = 120°$

The nice thing about using radians is that it makes certain formulas really simple. For example, in a circle of radius r with an interior angle of θ, the arclength, s, cut out by the angle (measured in radians) is given by:



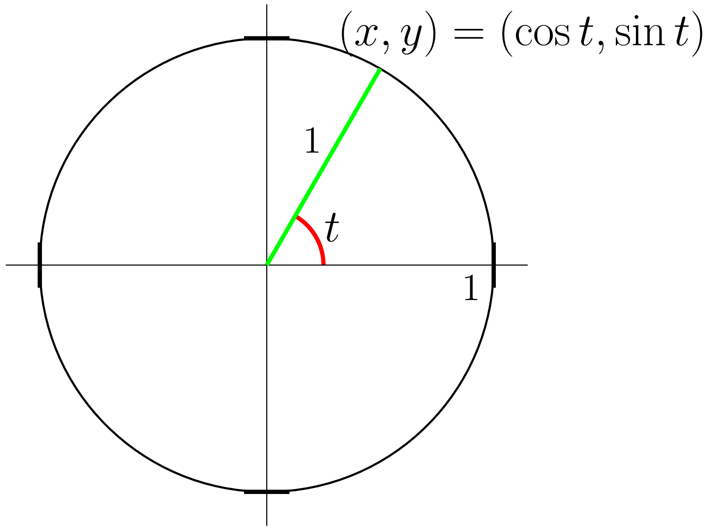
Ex: What is the arclengh cut from a circle of radius 5in if the interior angle is 4.2 radians?

$$s = (5\text{in})(4.2) = 21\text{in}$$

See the last page for the unit circle and special angles.

Trigonometric Functions

Def: An angle of t radians is measured counterclockwise around the unit circle from the positive x-axis, and intersects the unit circle at a point (x, y). We define $\cos(t) = x$ and $\sin(t) = y$.



The general form for a sine or cosine function is:

$$y = A\sin(B(x - C)) + D \qquad \text{or} \qquad y = A\sin(B(x - C)) + D$$

$|A|$ is the amplitude, defined to be half the distance between the maximum and minimum values.

$\frac{2\pi}{|B|}$ is the period, defined to be the time it takes for the function to complete one cycle. Functions that repeat over a fixed time interval are called periodic.

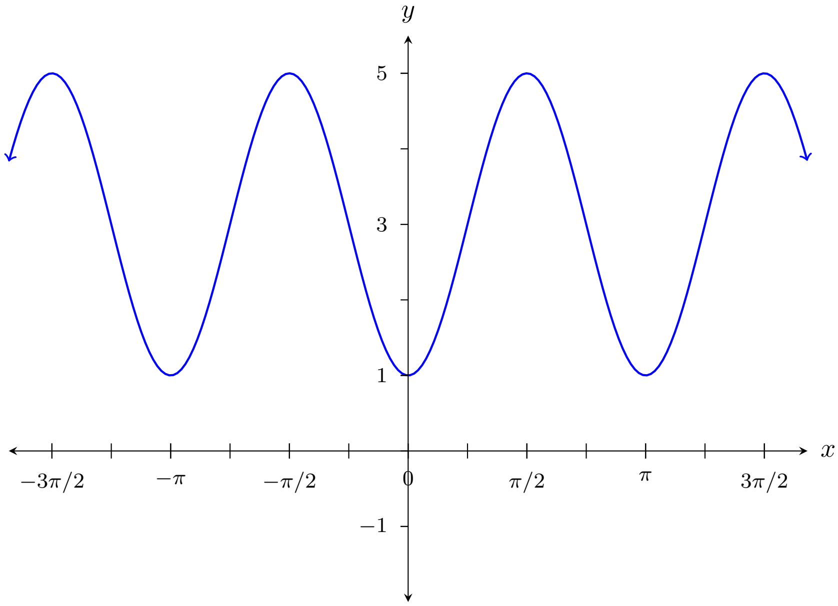
C is the phase shift, a horizontal shift for a trig function. $C > 0$ is a shift to the right. While $C < 0$ is a shift to the left.

Note: Sine and cosine differ by a phase shift of $\frac{\pi}{2}$. This gives us the idenities:

$$\cos(t) = \sin(t + \frac{\pi}{2}) \qquad \text{and} \qquad \sin(t) = \cos(t - \frac{\pi}{2})$$

D is the vertical shift, which is the average of the maximum and minimum values.

Ex: Determine an equation for the sinusoidal function below:



This function has a maximum of 5 and a minimum of 1, so the amplitude is

$$A = \frac{5-1}{2} = 2$$

.

The vertical shift is half-way between the maximum and minimum, so

$$D = \frac{5+1}{2} = 3$$

.

This function completes one full period every π units, so

$$\begin{aligned} period &= \frac{2\pi}{B} \\ \pi &= \frac{2\pi}{B} \\ B &= \frac{2\pi}{\pi} \\ B &= 2 \end{aligned}$$

Lastly, we need to determine the phase shift. This depends on which choice of function (sine or cosine) we use to model the graph. If we choose sine, then notice that since sine usually starts at $(0, 0)$ and increases to the right, then the next closest starting point for this function will be $(\frac{\pi}{4}, 3)$. This means the graph is shifted to the right by $\frac{\pi}{4}$ and thus:

$$C = \frac{\pi}{4}$$

And our equation is:

$$y = 2\sin\left(2\left(x - \frac{\pi}{4}\right)\right) + 3$$

Alternatively if we choose cosine, which starts at a maximum and then decreases to the right, we may choose as the starting point $(\frac{\pi}{2}, 5)$. This corresponds to a right-shift of $\frac{\pi}{2}$, and so here:

$$C = \frac{\pi}{2}$$

And our equation is:

$$y = 2\cos\left(2\left(x - \frac{\pi}{2}\right)\right) + 3$$

<u>Bonus:</u> Check that this equation will also work:

$$y = -2\cos(2x) + 3$$

<u>Def:</u> The tangent function is defined to be $\tan(t) = \frac{\sin(t)}{\cos(t)}$, and has a period of π. Also notice that $\tan(t)$ is undefined whereever $\cos(t) = 0$.

The general tangent function has the form:

$$y = A\tan(B(x - C)) + D$$

Where now the period is given by: $period = \frac{\pi}{|B|}$

Values for common reference angles on the unit circle are provided at the end of these notes. Using this information and the general definitions of trig functions we can solve basic problems involving trig functions.

<u>Ex:</u> Find all solutions to the equation: $2\sin(3x) = -\sqrt{2}$ for $0 \leq x \leq 2\pi$.

$$2\sin(3x) = -\sqrt{2}$$

$$\sin(3x) = -\frac{\sqrt{2}}{2}$$

$$3x = \frac{5\pi}{4} + 2\pi n \qquad 3x = \frac{7\pi}{4} + 2\pi n$$

$$x = \frac{5\pi}{12} + \frac{2}{3}\pi n \qquad x = \frac{7\pi}{12} + \frac{2}{3}\pi n$$

$$x = \frac{5\pi}{12} + \frac{8}{12}\pi n \qquad x = \frac{7\pi}{12} + \frac{8}{12}\pi n$$

$$x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}, \frac{21\pi}{12}, \frac{23\pi}{12}$$

Reciprocal trig functions are commonly refered to as secant, cosecant, and cotangent, and are defined by:

- Secant
  $\sec(x) = \frac{1}{\cos(x)}$
- Cosecant
  $\csc(x) = \frac{1}{\sin(x)}$
- Cotangent
  $\cot(x) = \frac{1}{\tan(x)}$

The following trig identities are used extensively in mathematics:

$$\sin^2(x) + \cos^2(x) = 1 \qquad \text{and} \qquad \tan^2(x) + 1 = \sec^2(x)$$

<u>Ex:</u> Simplify the expression: $\dfrac{\sin\theta}{1+\cos\theta} + \cot\theta$.

$$\begin{aligned}
\frac{\sin\theta}{1+\cos\theta} + \cot\theta &= \frac{\sin\theta}{1+\cos\theta} + \frac{\cos\theta}{\sin\theta} \\
&= \frac{\sin\theta}{1+\cos\theta}\cdot\frac{\sin\theta}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} \\
&= \frac{\sin^2\theta + \cos\theta + \cos^2\theta}{(1+\cos\theta)(\sin\theta)} \\
&= \frac{1+\cos\theta}{(1+\cos\theta)(\sin\theta)} \\
&= \frac{1}{\sin\theta} \\
&= \csc\theta
\end{aligned}$$

Inverse Trig Functions

Recall that a function is invertible if and only if it passes the horizontal line test on the given domain. Since trig functions do not pass the horizontal line test over the real numbers, we must restrict their domains in order to create inverse functions. For a given trig function we add the prefix "arc" to denote the inverse function.

- $\sin(x)$
  Restriction
  Domain: $[-\frac{\pi}{2}, \frac{\pi}{2}]$
  Range: $[-1, 1]$
- $\cos(x)$
  Restriction
  Domain: $[0, \pi]$
  Range: $[-1, 1]$
- $\tan(x)$
  Restriction
  Domain: $(-\frac{\pi}{2}, \frac{\pi}{2})$
  Range: $(-\infty, \infty)$
- $\arcsin(x)$
  Domain: $[-1, 1]$
  Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\arccos(x)$
  Domain: $[-1, 1]$
  Range: $[0, \pi]$
- $\arctan(x)$
  Domain: $(-\infty, \infty)$
  Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$

For normal trig functions you input an *angle* and get back *ratio* . For "arc" functions you input a *ratio* and get back an *angle*!

Ex: $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ so $\arcsin(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$, $\cos(0) = 1$ so $\arccos(1) = 0$,
$\tan(-\frac{\pi}{3}) = -\sqrt{3}$ so $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$,
BUT $\sin\frac{5\pi}{6} = \frac{1}{2}$ while $\arcsin(\frac{1}{2}) = \frac{\pi}{6}$!

Note: Sometimes you will see "arc" functions written with a "-1" before the argument. DO NOT CONFUSE THIS WITH THE RECIPROCAL! $\arcsin(x) = \sin^{-1}(x) \neq \frac{1}{\sin(x)} = \csc(x)$

Use the following diagram of the unit circle as reference for solving trig problems. It is strongly recommended that you have this memorized.