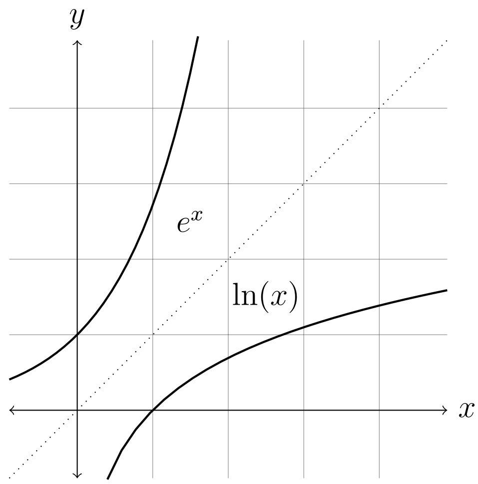
Math 1300 Section 1.4: Logarithmic Functions

1. Properties of Logarithms:

   (a) Definition: $\log_b(a) = x$ is exactly the same as $b^x = a$ for $b > 1$.
   Notation: $\ln(x) = \log_e(x)$ and $\log(x) = \log_{10}(x)$.

   (b) The function $\log_b(x)$ is the inverse of the function b^x. In the case $b = e$, we have the following picture

   (c) Domain of Log: The function $\log_b(x)$ is only defined for positive numbers. Hence the domain of $\log_b(x)$ is $(0, \infty)$.

   (d) For any $b > 1$, we have the following properties:

$$
\begin{aligned}
\log_b(AB) &= \log_b(A) + \log_b(B) & \ln(AB) &= \ln(A) + \ln(B) \\
\log_b(A/B) &= log_b(A) - \log_b(B) & \ln(A/B) &= \ln(A) - \ln(b) \\
\log_b(A^n) &= n \cdot \log_b(A) & \ln(A^n) &= n \cdot \ln(A) \\
\log_b(b^x) &= x & \ln(e^x) &= x \\
b^{\log_b(x)} &= x & e^{\ln(x)} &= x \\
\log_b(1) &= 0 & \ln(1) &= 0 \\
\log_b(b) &= 1 & \ln(e) &= 1
\end{aligned}
$$

   (e) Change of Base Formula: for any $b > 1$ and $c > 1$, we have

$$\log_b(x) = \frac{\log_c(x)}{\log_c(b)}$$

2. Solving Equations Using Logarithms:

   (a) Problem: Solve $\log_2(8) = x$ for x.
   Solution: Since $\log_2(8) = x$ is the same statement as $8 = 2^x$, we need only know how many 2's make up 8. Since $2^3 = 8$, it must be that $x = 3$.

   (b) Problem: Solve $3^x = 5$ for x.
   Solution: We want to remove the variable from the exponent, so we want to take a logarithm of both sides. Since the base is a 3, we may as well use log base 3. Solving,

$$\begin{aligned} 3^x &= 5 && \text{initial equation} \\ \log_3(3^x) &= \log_3(5) && \text{take } \log_3 \text{ of both sides} \\ x &= \log_3(5) && \log_3 \text{ ``undoes'' raising 3 to anything} \end{aligned}$$

   (c) Problem: Solve $5^t = e \cdot 7^t$ for t.
   Solution:

$$\begin{aligned} 5^t &= e \cdot 7^t && \text{initial expression} \\ \ln(5^t) &= \ln(e \cdot 7^t) && \text{take ln of both sides} \\ \ln(5^t) &= \ln(e) + \ln(7^t) && \text{property of logs - multiplication to addition} \\ t \cdot \ln(5) &= \ln(e) + t \cdot \ln(7) && \text{property of logs - bring down exponent} \\ t \cdot \ln(5) - t \cdot \ln(7) &= \ln(e) && \text{move all terms involving } t \text{ to one side} \\ t \cdot (\ln(5) - \ln(7)) &= \ln(e) && \text{factor out the } t \\ t \cdot (\ln(5) - \ln(7)) &= 1 && \text{ln of } e \text{ is 1} \\ t &= \frac{1}{(\ln(5) - \ln(7))} && \text{divide by } (\ln(5) - \ln(7)) \text{ because it is nonzero} \end{aligned}$$

   (d) Problem: Solve $e^{2y-3} = 1$ for y.
   Solution:

$$\begin{aligned} e^{2y-3} &= 1 && \text{inital problem} \\ \ln\left(e^{2y-3}\right) &= \ln(1) && \text{take ln of both sides} \\ 2y - 3 &= \ln(1) && \text{ln ``undoes'' raising } e \text{ to anything} \\ 2y - 3 &= 0 && \text{ln of 1 is 0} \\ 2y &= 3 && \text{move } y\text{'s to one side} \\ y &= \frac{3}{2} && \text{divide away the 2} \end{aligned}$$

(e) Problem: Solve $\log_2(x+2) + \log_2(x-2) = \log_2(5)$ for x.
Solution:

$$\begin{aligned}
\log_2(x+2) + \log_2(x-2) &= \log_2(5) && \text{inital problem}\\
\log_2((x+2)(x-2)) &= \log_2(5) && \text{property of logs - addition to multiplication}\\
2^{\log_2((x+2)(x-2))} &= 2^{\log_2(5)} && \text{raise 2 to both sides}\\
(x+2)(x-2) &= 5 && \log_2 \text{ and exponentiation by 2 are inverses}\\
x^2 - 4 &= 5 && \text{multiply out polynomials}\\
x^2 &= 9 && \text{move } x\text{'s to one side}\\
x &= \pm 3 && \text{solve for } x
\end{aligned}$$

Note that $x = -3$ is not a solution to the problem because if we plug it in, we get the expression $\log_2((-3)+2)+\log_2((-3)-2) = \log_2(5)$ but $\log_2(-1)$ and $\log_2(-5)$ are not defined because we can't take any kind of log of a negative number! Hence our final answer is $x = 3$.

(f) Problem: Solve $e^{2z} - 4e^z + 4 = 0$ for z.
Solution: Notice that e^{2z} is the same thing as $(e^z)^2$ and make the substituation $u = e^z$.

$$\begin{aligned}
e^{2z} - 4e^z + 4 &= 0 && \text{inital problem}\\
(e^z)^2 - 4e^z + 4 &= 0 && \text{property of exponentials - powers}\\
u^2 - 4u + 4 &= 0 && \text{substitute } u \text{ for } e^z\\
(u-2)(u-2) &= 0 && \text{factor}\\
u &= 2 && \text{solve for u}\\
e^z &= 2 && \text{switch } u \text{ out for original } e^z\\
\ln(e^z) &= \ln(2) && \text{take ln of both sides}\\
z &= \ln(2) && \text{ln ``undoes'' raising } e \text{ to anything}
\end{aligned}$$

(g) Problem: Solve $e^{2z} - 2e^z - 3 = 0$ for z.
Solution: By substituting $u = e^z$, we have

$$\begin{aligned}
e^{2z} - 2e^z - 3 &= 0 && \text{inital problem}\\
(e^z)^2 - 2e^z - 3 &= 0 && \text{property of exponentials - powers}\\
u^2 - 2u - 3 &= 0 && \text{substitute } u \text{ for } e^z\\
(u-3)(u+1) &= 0 && \text{factor}\\
u &= 2, -1 && \text{solve for u}
\end{aligned}$$

Note that $u = -1$ cannot be a solution to the problem because $u = e^z > 0$ for any z. This is the same kind of restriction as was in part (g) of logarithmic equation, which ultimately arise from the domain of logarithmic function and the range of exponential function. Hence, from $u = 2$, i.e., $e^z = 2$, we get $z = \ln 2$ as our solution for the initial equation. Note that our solution for this problem is z, not u.