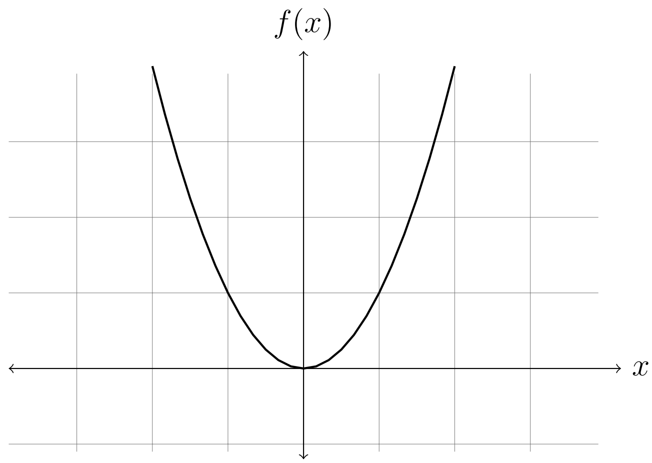
Math 1300 Section 1.3: New Functions From Old

1. Shifts, Stretches, Flips:

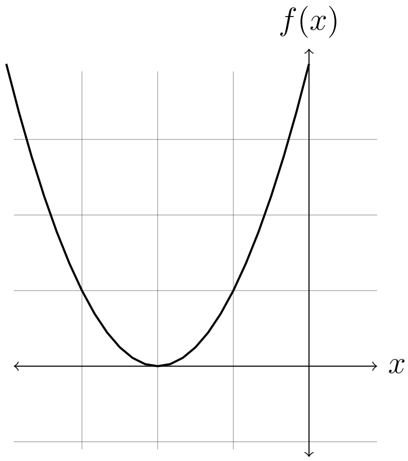
   (a) If $f(x)$ is a function, we can perform the following operations:

$c \cdot f(x)$	vertical stretch by c if $c > 1$
$c \cdot f(x)$	vertical compression by c if $0 < c < 1$
$f(c \cdot x)$	horizontal compression by c if $c > 1$
$f(c \cdot x)$	horizontal stretch by c if $0 < c < 1$
$f(x) + a$	move up by a
$f(x - a)$	move *right* by a
$-f(x)$	vertical flip about x-axis
$f(-x)$	horizontal flip about y-axis

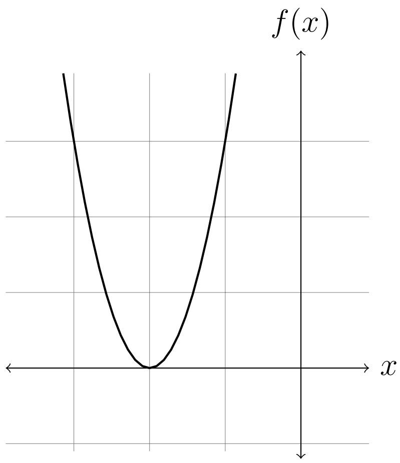
Example: $f(x) = -3(x+2)^2 + 5$



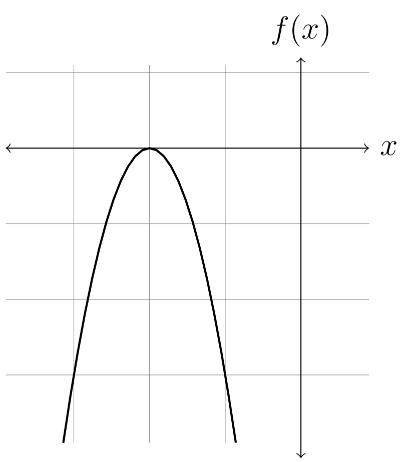
We start with a function we know: $y = x^2$



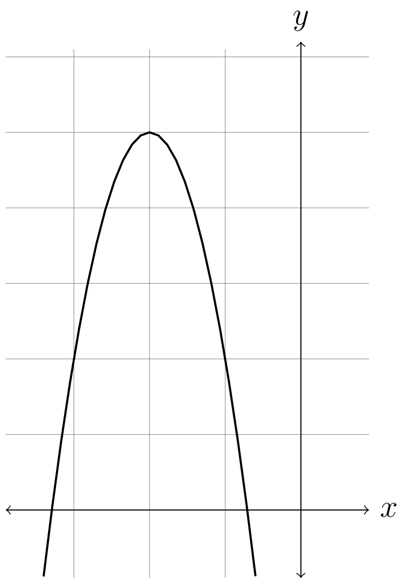
Shift 2 to the left: $y = (x + 2)^2$



Stretch vertically by 3: $y = 3(x + 2)^2$



Flip over x-axis: $y = -3(x + 2)^2$



Shift up by 5: $y = -3(x+2)^2 + 5$

(b) Definition: a function f is **odd** if $f(-x) = -f(x)$ (analytically). This is the same as f being symmetric about the origin (graphically).

Examples: x, x^3, $\sin(x)$

Definition: a function f is **even** if $f(-x) = f(x)$ (analytically). This is the same as f being symmetric about the y-axis (graphically).

Examples: x^2, $|x|$, $\cos(x)$

Problem: Is $f(x) = x^3 + x$ even, odd, or neither?

Solution: Plug in $-x$ everywhere there is an x in the function and simplify. If we get exactly the original function back, the function is even. If we get exactly the negative, it is odd. If neither happens, the function is neither even nor odd.

$$\begin{aligned}
f(-x) &= (-x)^3 + (-x) && \text{plug in } -x \text{ for } x \\
&= -(x^3) - x && \text{negatives come out from under odd exponents} \\
&= -(x^3 + x) && \text{pulling out the negatives} \\
&= -f(x) && \text{remembering what } f(x) \text{ was to start with}
\end{aligned}$$

Since we got exactly the negative of what we started with, $f(x)$ must be **odd**.

Problem: Is $f(x) = x^2 + x^3$ even, odd, or neither?

Solution: Again, we must check what happens when we plug in $-x$ for x everywhere in the function.

$$\begin{aligned} f(-x) &= (-x)^2 + (-x)^3 && \text{plug in } -x \text{ for } x \\ &= x^2 + (-x)^3 && \text{negatives cancel under even exponents} \\ &= x^2 - x^3 && \text{negatives come out from under odd exponents} \end{aligned}$$

Since this is neither exactly what we started with nor its opposite, our function is **neither** even nor odd.

2. Composite Functions:

   (a) To compose two functions $f(x)$ and $g(x)$, we can make new functions by taking one and plugging it in anywhere x appears in the other.

   Example: if $f(x) = x^2 + 1$ and $g(x) = x - 2$, we can form

   $$f(g(x)) = (x-2)^2 + 1 = (x^2 - 4x + 4) + 1 = x^2 - 4x + 5$$

   and

   $$g(f(x)) = (x^2 + 1) - 2 = x^2 - 1$$

   (b) To decompose a function into a composition of functions, we identify an obvious "outside" and an obvious "inside" function.

   Example: If $h(x) = \sqrt{\tan(x)}$, we can see that we are taking the square root of values of a tangent function, so we can represent $h(x)$ as $h(x) = g(f(x))$ where $f(x) = \tan(x)$ and $g(x) = \sqrt{x}$.

3. Inverse Functions:

   Definition: Two functions $f(x)$ and $g(x)$ are **inverses** to one another if $g(f(x)) = x$ and $f(g(x)) = x$ (analytically). This is the same as $f(x)$ and $g(x)$ being reflections of one another across the line $y = x$ (graphically). The inverse of $f(x)$ is usually denoted $f^{-1}(x)$.

   Note: The domain of $f(x)$ is the range of $f^{-1}(x)$ and vice versa.

   Existence: (graphically) $f(x)$ has an inverse if it passes the horizontal line test (note the vertical line test is used to ensure a graph represents a function in the first place).

   Example: $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses because $\sqrt[3]{x^3} = x$ and $(\sqrt[3]{x})^3 = x$.

Problem: Find the inverse to $y = f(x) = 6x - 2$.

Solution: To find an inverse a function $y = f(x)$ (analytically), we switch x and y, getting an expression $x = f(y)$ then solve for y.

$$\begin{aligned} x &= 6y - 2 && \text{switch } x \text{ and } y \\ 6y &= x + 2 && \text{move all } y\text{'s to one side of equation} \\ y &= \frac{x+2}{6} && \text{simplify} \\ f^{-1}(x) &= \frac{x+2}{6} && \text{final answer} \end{aligned}$$

Check:

$$f(f^{-1}(x)) = f(\frac{x+2}{6}) = 6(\frac{x+2}{6}) - 2 = x$$

and

$$f^{-1}(f(x) = f^{-1}(6x-2) = \frac{(6x-2)+2}{6} = \frac{6x}{6} = x$$

so $f(x)$ and $f^{-1}(x)$ are inverse functions.