1. Section 1.2 of HH - Exponential Functions

   (a) Exponential Functions: An *exponential function* has the form $y = ab^x$, or using function notation, $f(x) = ab^x$. Note that the variable, x, is in the exponent.

   - a is the initial value, the value of the function when $x = 0$, because $b^0 = 1$.
   - b must be a positive real number. If $b > 1$, then the relationship is called exponential growth. If $0 < b < 1$, then the relationship is called exponential decay.
   - Just as linear functions have a constant rate of change, exponential functions have a constant percent change. For example, if a population increases by 5% every year, then the growth factor b is computed to be $1 + .05 = 1.05$. If the population decreases by 5% every year, then the decay factor b is computed to be $1 - .05 = .95$.

   (b) The Family of Exponential Functions: Changing the values of a and b in an exponential function changes the graph of that function. The value of a just determines the y-intercept of the graph. The value of b determines whether the graph increases or deacreases, and how quickly it increases or decreases. Refer to Figures 1.19 and 1.20 in section 1.2 to review these patterns.

   (c) Continuous Growth/Decay: Every exponential function $f(x) = ab^x$ can be written in the form $f(x) = ae^{kx}$ (in this form, $b = e^k$ since $f(x) = ae^{kx} = a(e^k)^x$).

   - If k is a positive number, then $e^k > 1$ so this is continuous growth.
   - If k is a negative number, then $0 < e^k < 1$, so this is continuous decay.
   - The positive number $|k|$ is the continuous rate of growth/decay. It is important to be able to convert exponential functions between these two forms. For example, if you were asked for the growth rate of the exponential function whose continuous growth rate is 50%, then $f(x) = ae^{.5x} = a(e^{.5})^x = a \cdot 1.65^x$ shows that the growth rate is 1.65 (so there is a constant growth of 65%).

   (d) Half - life/Doubling - time:

   - The half-life of a substance is the amount of time it takes for the initial amount to be cut in half. If the half-life of a substance is t_0 and the initial amount is a, then the exponential function modeling this situation is $f(t) = a(\frac{1}{2})^{\frac{t}{t_0}}$ (note that $f(t_0) = \frac{1}{2}a$, which is the definition of half-life). For example, if the half-life of a radioactive substance was 2 years, then the amount of this substance left after time t would be $f(t) = a(\frac{1}{2})^{\frac{t}{2}} = a(.71)^t$. So the decay factor is .71 (meaning the substance is decreasing by 29% each year).
   - The doubling time of a substance is the amount of time it takes for the initial amount to be doubled. If the doubling time of a substance is t_0 and the initial amount is a, then the exponential function modeling this situation is $f(t) = a(2)^{\frac{t}{t_0}}$ (note that $f(t_0) = 2a$, which is the definition of doubling time).

For example, if the doubling time of a population was 2 years, then population after time t would be $f(t) = a(2)^{\frac{t}{2}} = a(1.41)^t$. So the growth factor is 1.41 (meaning the population is increasing by 41% each year).

(e) Concavity: Intuitively, concavity describes the way a graph bends on a given interval. If the graph bends upward from left to right on an interval ($\cup$), we say it is concave up on that interval. If the graph bends downward on an interval ($\cap$), we say it is concave down on that interval. For example, exponential functions are concave up everywhere: exponential growth functions resemble the right half of $\cup$, while exponential decay functions resemble the left half of $\cup$.

2. Examples

(a) Suppose the value of a brand new car is $25,000$ dollars. Write a function for the value of the car after t years if

- the value of the car depreciates 300 dollars every year.

  Since the dependent variable (value of the car) is changing by a constant amount every year, this is describing a linear relationship. Since the initial value is $25,000$ dollars and the value decreases 300 dollars per year, the function describing this relationship is

$$A(t) = -300t + 25,000.$$

- the value of the car depreciates 12% every year.

  Since the dependent variable (value of the car) is changing by a constant percent every year, this is describing an exponential relationship. Since the initial value is $25,000$ dollars and the value decreases 12% every year (so the decay factor is $1 - .12 = .88$), the function describing this relationship is

$$A(t) = 25,000(.88)^t.$$

(b) Suppose that $g(x)$ is an exponential function with $g(2) = 4$ and $g(5) = 20$. Write the equation for $g(x)$.

Solution. Since $g(x)$ is exponential, we can write $4 = ab^2$ and $20 = ab^5$. Then taking the ratio, we get

$$\frac{20}{4} = \frac{ab^5}{ab^2},$$

so the a's cancel and we get $5 = b^3$, so $b = \sqrt[3]{5}$. Now we can plug b back into either equation above to solve for a:

$$4 = a(\sqrt[3]{5})^2,$$

so $a = \frac{4}{5^{\frac{2}{3}}}$. Since $\sqrt[3]{5} \approx 1.71$ and $\frac{4}{5^{\frac{2}{3}}} \approx 1.37$, we have

$$g(x) = 1.37(1.71)^x.$$

(c) A population grows by 4% every month. By what percent does this population increase each year? Each day (assume 30 days in a month)?

Solution. The growth factor per month is 1.04. Since there are 12 months in a year, the population will have increased by a factor of $1.04^{12} = 1.6$ after one year. Therefore, the population grows by 60% each year.

If b is the growth rate per day, since there are 30 days in a month, we would have $b^{30} = 1.04$. So $b = 1.04^{\frac{1}{30}} = 1.0013$. Therefore, the population grows by .13% each day.