Math 1300-002: L'Hôpital's Rule Practice

Compute the following limits using l'Hôpital's Rule:

- $\displaystyle\lim_{x\to\infty} \frac{7x^2 - 10x + 1}{3x^2 + 5}$

  This limit has the form $\frac{\infty}{\infty}$, so we can apply L'Hôpital's Rule directly:

$$\begin{aligned}
\lim_{x\to\infty} \frac{7x^2 - 10x + 1}{3x^2 + 5} &\overset{\text{H}}{=} \lim_{x\to\infty} \frac{14x - 10}{6x} && \left(\text{form: } \frac{\infty}{\infty}\right) \\
&\overset{\text{H}}{=} \lim_{x\to\infty} \frac{14}{6} \\
&= \frac{7}{3}.
\end{aligned}$$

- $\displaystyle\lim_{x\to 0^-} \left(\frac{3}{x} - \frac{1}{e^x - 1}\right)$

  This limit has form $\infty - \infty$, so we rearrange it by finding a common denominator:

$$\begin{aligned}
\lim_{x\to 0^-} \left(\frac{3}{x} - \frac{1}{e^x - 1}\right) &= \lim_{x\to 0^-} \frac{3(e^x - 1) - x}{x(e^x - 1)} && \left(\text{form: } \frac{0}{0}\right) \\
&\overset{\text{H}}{=} \lim_{x\to 0^-} \frac{3e^x - 1}{xe^x + (e^x - 1)} \\
&= \frac{2}{0^-} = -\infty.
\end{aligned}$$

- $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right)$

  This limit has form $\infty - \infty$, so we rearrange it by finding a common denominator:

$$\begin{aligned}
\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right) &= \lim_{x\to 0} \frac{\sin(x) - x}{x\sin(x)} && \left(\text{form: } \frac{0}{0}\right) \\
&\overset{\text{H}}{=} \lim_{x\to 0} \frac{\cos(x) - 1}{x\cos(x) + \sin(x)} && \left(\text{form: } \frac{0}{0}\right) \\
&\overset{\text{H}}{=} \lim_{x\to} \frac{-\sin(x)}{-x\sin(x) + \cos(x) + \cos(x)} \\
&= \frac{0}{2} = 0.
\end{aligned}$$

- $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$

  This limit has form $\infty - \infty$, so we rearrange it by finding a common denominator:

$$\begin{aligned}
\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right) &= \lim_{x\to 0} \frac{e^x - 1 - x}{x(e^x - 1)} && \left(\text{form: } \frac{0}{0}\right) \\
&\overset{\text{H}}{=} \lim_{x\to 0^-} \frac{e^x - 1}{xe^x + (e^x - 1)} && \left(\text{form: } \frac{0}{0}\right) \\
&\overset{\text{H}}{=} \lim_{x\to 0^-} \frac{e^x}{xe^x + e^x + e^x} \\
&= \frac{1}{2}.
\end{aligned}$$

- $\displaystyle\lim_{x\to\infty} \frac{\ln(3x)}{5\ln(2x+1)}$

  This limit has the form $\frac{\infty}{\infty}$, so we can apply L'Hôpital's Rule directly:

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(3x)}{5\ln(2x+1)} &\overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{3x}\cdot 3}{\frac{5}{2x+1}\cdot 2} \\ &= \lim_{x\to\infty} \frac{2x+1}{10x} && \left(\text{form: } \frac{\infty}{\infty}\right) \\ &= \lim_{x\to\infty} \frac{2}{10} = \frac{1}{5}.\end{aligned}$$

- $\displaystyle\lim_{x\to\pi^+} \sin(x)\ln(x-\pi)$

  This limit has form $0\cdot(-\infty)$, so we rearrange it by forcing it into a fraction:

$$\begin{aligned}\lim_{x\to\pi^+} \sin(x)\ln(x-\pi) &= \lim_{x\to\pi^+} \frac{\ln(x-\pi)}{\frac{1}{\sin(x)}} && \left(\text{form: } \frac{-\infty}{\infty}\right) \\ &\overset{\text{H}}{=} \lim_{x\to\pi^+} \frac{\frac{1}{x-\pi}}{\frac{-\cos(x)}{\sin^2(x)}} \\ &= \lim_{x\to\pi^+} \frac{-\sin^2(x)}{\cos(x)(x-\pi)} && \left(\text{form: } \frac{0}{0}\right) \\ &\overset{\text{H}}{=} \lim_{x\to\pi^+} \frac{-2\sin(x)\cos(x)}{\sin(x)(x-\pi)+\cos(x)} = \frac{0}{-1} = 0.\end{aligned}$$

- $\displaystyle\lim_{x\to\infty} x^{\frac{1}{x}}$

  This time, the form is ∞^0, so we'll use logs. Let $y = x^{\frac{1}{x}}$. Then, by the power rule for logs,

$$\begin{aligned}\lim_{x\to\infty} \ln(y) &= \lim_{x\to\infty} \frac{1}{x}\ln(x) = \lim_{x\to\infty} \frac{\ln(x)}{x} && \left(\text{form: } \frac{\infty}{\infty}\right) \\ &\overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.\end{aligned}$$

  By the continuity of $\ln(x)$, we can exponentiate both sides to obtain $\displaystyle\lim_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1$.

- $\displaystyle\lim_{x\to 6^-} (x-5)^{\frac{x-2}{x-6}}$

  The form is $1^{(-\infty)}$, so we'll use logs. Let $y = (x-5)^{\frac{x-2}{x-6}}$. Then, by the power rule for logs,

$$\begin{aligned}\lim_{x\to 6^-} \ln(y) &= \lim_{x\to 6^-} \frac{x-2}{x-6}\ln(x-5) = \lim_{x\to 6^-} \frac{(x-2)\ln(x-5)}{x-6} && \left(\text{form: } \frac{\infty}{\infty}\right) \\ &\overset{\text{H}}{=} \lim_{x\to 6^-} \frac{(x-2)\frac{1}{x-5}(1) + (1)\ln(x-5)}{1} = 4.\end{aligned}$$

  By the continuity of $\ln(x)$, we can exponentiate both sides to obtain $\displaystyle\lim_{x\to\infty} (x-5)^{\frac{x-2}{x-6}} = e^4$.