1. Evaluate the following limits.

(a) $\displaystyle\lim_{h\to 0}\frac{(\sqrt{2}+h)^2-2}{h}$

Solution. We have

$$\lim_{h\to 0}\frac{(\sqrt{2}+h)^2-2}{h}=\lim_{h\to 0}\frac{(\sqrt{2}+h)^2-(\sqrt{2})^2}{h}=\lim_{h\to 0}\frac{(\sqrt{2}+h-\sqrt{2})(\sqrt{2}+h+\sqrt{2})}{h}$$
$$=\lim_{h\to 0}\frac{h(\sqrt{2}+h+\sqrt{2})}{h}=\lim_{h\to 0}(\sqrt{2}+h+\sqrt{2})=2\sqrt{2}.$$

(b) $\displaystyle\lim_{h\to 0}\frac{(2+h)^3-8}{h}$

Solution. We have

$$\lim_{h\to 0}\frac{(2+h)^3-8}{h}=\lim_{h\to 0}\frac{(2+h)^3-2^3}{h}=\lim_{h\to 0}\frac{(2+h-2)((2+h)^2+2(2+h)+4)}{h}$$
$$=\lim_{h\to 0}\frac{h((2+h)^2+2(2+h)+4)}{h}=\lim_{h\to 0}((2+h)^2+2(2+h)+4)$$
$$=2^2+2(2)+4=12.$$

(c) $\displaystyle\lim_{t\to 0}\left(\frac{1}{t\sqrt{t+1}}-\frac{1}{t}\right)$

Solution. We have

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{t+1}}-\frac{1}{t}\right)=\lim_{t\to 0}\left(\frac{1-\sqrt{1+t}}{t\sqrt{t+1}}\right)=\lim_{t\to 0}\left(\frac{(1-\sqrt{1+t})(1+\sqrt{1+t})}{t\sqrt{t+1}(1+\sqrt{1+t})}\right)$$
$$=\lim_{t\to 0}\left(\frac{(1-(1+t))}{t\sqrt{t+1}(1+\sqrt{1+t})}\right)=\lim_{t\to 0}\left(\frac{-t}{t\sqrt{t+1}(1+\sqrt{1+t})}\right)$$
$$=\lim_{t\to 0}\left(\frac{-1}{\sqrt{t+1}(1+\sqrt{1+t})}\right)=\frac{-1}{\sqrt{1}(1+\sqrt{1})}=-\frac{1}{2}.$$

(d) $\displaystyle\lim_{h\to 0}\left(\frac{\frac{1}{\sqrt{1+h}}-1}{h}\right)$

Solution. We have

$$\lim_{h\to 0}\left(\frac{\frac{1}{\sqrt{1+h}}-1}{h}\right)=\lim_{h\to 0}\left(\frac{1-\sqrt{1+h}}{h\sqrt{1+h}}\right)=-\frac{1}{2}\quad\text{(see the solution to }\mathbf{1(c)}\text{)}.$$

You should check these solutions carefully and justify each step that I've made. If there is something that you do not understand, please ask.

2. Show that $\lim_{x\to 0} \sqrt{x^3+x}\,\sin\left(\frac{\pi}{x}\right) = 0.$

Solution. Since $-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$ for all $x \ne 0$ and square roots are nonnegative,

$$-\sqrt{x^3+x} \le \sqrt{x^3+x}\,\sin\left(\frac{\pi}{x}\right) \le \sqrt{x^3+x}$$

for all $x > 0$. Note that taking left-hand limits does not make sense here, since $x^3+x<0$ for all $x<0$. Therefore, the right-hand limit and the limit coincide. Since

$$\lim_{x\to 0}\sqrt{x^3+x} = \sqrt{\lim_{x\to 0^+}(x^3+x)} = \sqrt{0} = 0,$$

we also have

$$\lim_{x\to 0} -\sqrt{x^3+x} = -\lim_{x\to 0}\sqrt{x^3+x} = 0.$$

Thus

$$\lim_{x\to 0}\sqrt{x^3+x}\,\sin\left(\frac{\pi}{x}\right) = 0$$

by the Squeeze Theorem.

3. Given $\lim_{x\to 2} f(x) = 4$, $\lim_{x\to 2} g(x) = -2$ and $\lim_{x\to 2} h(x) = 0$, find

(a) $\lim_{x\to 2}\sqrt{\big(f(x)\big)^3+1}$

Solution. We have

$$\lim_{x\to 2}\sqrt{\big(f(x)\big)^3+1} = \sqrt{\lim_{x\to 2}\big(f(x)\big)^3+1} = \sqrt{\left(\lim_{x\to 2} f(x)\right)^3+1} = \sqrt{4^3+1} = \sqrt{65}.$$

(b) $\lim_{x\to 2}\left(\frac{1}{\big(g(x)\big)^2} - \frac{1}{f(x)}\right)$

Solution. We have

$$\lim_{x\to 2}\left(\frac{1}{\big(g(x)\big)^2} - \frac{1}{f(x)}\right) = \lim_{x\to 2}\frac{1}{\big(g(x)\big)^2} - \lim_{x\to 2}\frac{1}{f(x)} = \frac{1}{\lim_{x\to 2}\big(g(x)\big)^2} - \frac{1}{\lim_{x\to 2} f(x)}$$
$$= \frac{1}{\left(\lim_{x\to 2} g(x)\right)^2} - \frac{1}{4} = \frac{1}{(-2)^2} - \frac{1}{4} = 0.$$

(c) $\lim_{x\to 2}\frac{g(x)\big(h(x)\big)^2}{f(x)h(x)}$

Solution. We have

$$\lim_{x\to 2}\frac{g(x)\big(h(x)\big)^2}{f(x)h(x)} = \left(\lim_{x\to 2}\frac{g(x)}{f(x)}\right)\left(\lim_{x\to 2}\frac{\big(h(x)\big)^2}{h(x)}\right)$$
$$= \left(\frac{\lim_{x\to 2} g(x)}{\lim_{x\to 2} f(x)}\right)\left(\lim_{x\to 2} h(x)\right) = \frac{-2}{4}\cdot 0 = 0.$$

(d) $\lim_{x \to 2} \sqrt[3]{4\,g(x)}$

Solution. We have

$$\lim_{x \to 2} \sqrt[3]{4\,g(x)} = \sqrt[3]{\lim_{x \to 2} 4\,g(x)} = \sqrt[3]{4 \lim_{x \to 2} g(x)} = \sqrt[3]{4(-2)} = \sqrt[3]{-8} = -2.$$