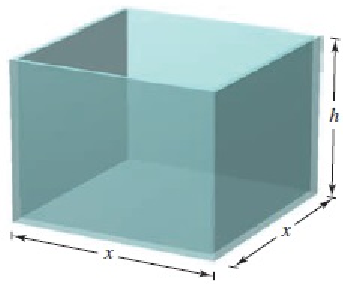
Name ___________________________

MATH 1300-015
Section 4.6

1. A rectangular box with a square base and no top is to be made of a total of 120cm^2 of cardboard. Find the dimension of the box of mimum volume.
(In this problem, assume that the volume is always positive.)
Answer:



Note that the objective function and the constraint equation are given by the following:

$$\begin{cases} V = x^2 h & \text{objective function} \\ S = x^2 + 4xy = 120 & \text{constraint equation} \end{cases}$$

From the constraint equation $x^2 + 4xy = 120$, we can eliminate x; since $4xy = 120 - x^2$ we get $y = \dfrac{120 - x^2}{4x}$. Hence,

$$V(x) = x^2 y = x^2 \left(\frac{120 - x^2}{4x} \right).$$

Now we need to find out the domain of V. First, $x > 0$ since x is a length. Note that x cannot be zero because that will make the volume zero. Second, from the constraint equation $4xy = 120 - x^2$, we know that

$$120 - x^2 > 0,$$

since $4xy$ represent the total area of the four sides of the cube and it must be positive. The solution of above inequality is $-\sqrt{120} < x < \sqrt{120}$. Combining these two inequalities, we get $0 < x < \sqrt{120}$. Hence, we obtain the following absolute minimum problem:

$$\text{find where the function } V(x) = x^2 \left(\tfrac{120 - x^2}{4x} \right) \text{ on } (0, \infty) \text{ attains the absolute minimum.}$$

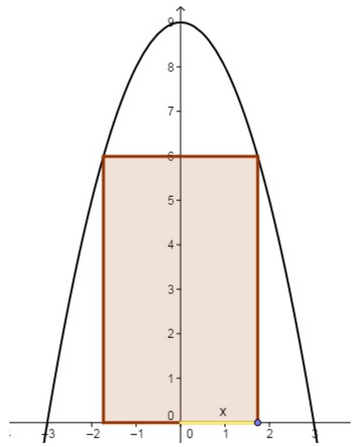
On its domain, the function V simplifies to the following:

$$V(x) = x^2 \left(\frac{120 - x^2}{4x} \right) = 30x - \frac{1}{4}x^3.$$

Then $V'(x) = 30 - \frac{3}{4}x^2$. Solving $V'(x) = 0$, we get $x = -\sqrt{40}$, $\sqrt{40}$. Hence, V has one critical number $\sqrt{40}$. Since $V''(x) = -\frac{3}{2}x$ and $V''(\sqrt{40}) = -\frac{3}{2} \cdot \sqrt{40} < 0$, V attains a local maximum at $x = \sqrt{40}$ by the Second Derivative Test. Since V has only one critical number and V attains a local maximum there, V must attain the absolute maximum at $x = \sqrt{40}$. From the constraint equation, the corresponding height is $h = \frac{20}{\sqrt{40}}$. Thus, the dimension of the box that gives the maximum volume is $\sqrt{40}\text{cm} \times \sqrt{40}\text{cm} \times \frac{20}{\sqrt{40}}\text{cm}$

2. A rectangle is incribed in the region bounded by the x-axis and the parabola $y = 9 - x^2$.
   (1) Find the height and width of the rectangle of greatest *area*.
   (2) Find the height and width of the rectangle of greatest *perimeter*.
   (3) Does the rectangle of greatest area have the greatest perimeter?

Answer:



(1) Let (x, y) be the point in the first quadrant which is on the parabola. Then the objective function and the constraint equation are given by the following:

$$\begin{cases} A = 2xy & \text{objective function} \\ y = 9 - x^2 & \text{constraint equation} \end{cases}$$

By substituting in the constraint equation to the objective function, we get

$$A(x) = 2x(9 - x^2) = 18x - 2x^3.$$

Now we need to find the domain of A. Since (x, y) is a point on the first quadrant, $0 \le x \le 3$. Hence, we obtain the following absolute maximum problem:

find where the function $A(x) = 18x - 2x^3$ on $[0, 3]$ attains the absolute maximum.

Note that

$$A'(x) = 18 - 6x^2 = -6(x - \sqrt{3})(x + \sqrt{3}).$$

Solving $A'(x) = 0$, we get $x = -\sqrt{3},\ \sqrt{3}$. Hence, A has one critical number $\sqrt{3}$. Note that

x	$A(x)$	
0	0	
$\sqrt{3}$	$12\sqrt{3}$	largest
3	0	

Hence, by the Closed Interval Method, the function A attains the absolute maximum at $x = \sqrt{3}$. The corresponding y-coordinate is $y = 6$. Thus, the height and width of the rectangle of greatest area are 6 and $2\sqrt{3}$, respectively.

(2) In this case, the objective function and the constraint equation are given by the following:

$$\begin{cases} P = 4x + 2y & \text{objective function} \\ y = 9 - x^2 & \text{constraint equation} \end{cases}$$

By substituting in the constraint equation to the objective function, we get

$$P(x) = 4x + 2(9 - x^2) = -2x^2 + 4x + 18.$$

As in the previous case, the domain is $[0, 3]$ and we obtain the following absolute maximum problem:

find where the function $P(x) = -2x^2 + 4x + 18$ on $[0, 3]$ attains the absolute maximum.

Note that

$$P'(x) = -4x + 4.$$

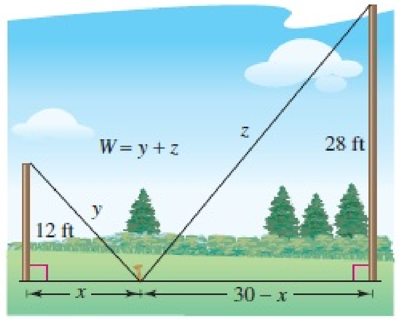
Solving $P'(x) = 0$, we get $x = 1$. Hence, A has one critical number 1. Note that

x	$P(x)$	
0	0	
1	20	largest
3	0	

Hence, by the Closed Interval Method, the function P attains the absolute maximum at $x = 1$. The corresponding y-coordinate is $y = 8$. Thus, the height and width of the rectangle of greatest area are 8 and 2, respectively.

(3) No.

3. Two posts, one 12 feet high and the other 28 feet high, stand 30 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each other post. Where should the stake be placed to use the least amount of wire?
Answer:



Let y and z be the lengths of two hypotenuse as in the picture. Let x be the distance between the left post and the stake. Then, by the Pythagorean Theorem, we get the following two constraint equations:

$$x^2 + 12^2 = y^2,$$
$$(30 - x)^2 + 28^2 = z^2.$$

So the objective function and the constraint equations are the following:

$$\begin{cases} W = y + z & \text{objective function} \\ x^2 + 12^2 = y^2,\ (30 - x)^2 + 28^2 = z^2 & \text{constraint equation} \end{cases}$$

We want to write W as a function of x. Note that x must satisfy $0 \leq x \leq 30$. From the constrant equations we can eliminate the two variables y and z; by solving for y and z, we get

$$y = \sqrt{x^2 + 144},$$
$$z = \sqrt{x^2 - 60x + 1684}.$$

Here, we only take the positive square root since y and z are lengths. So W is given by

$$W = y + z = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}.$$

Hence, we obtain the following absolute maximum problem:

find where the function $W(x) = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}$ on $[0, 30]$ attains the absolute minimum.

Note that

$$W'(x) = \frac{x}{\sqrt{x^2 + 144}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}}.$$

Setting $W'(x) = 0$, we get

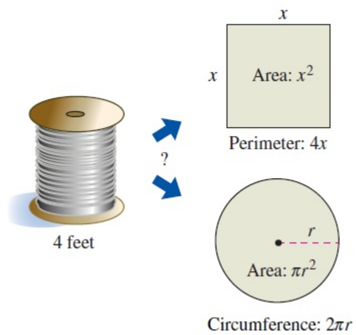
$$\begin{aligned}
\frac{x}{\sqrt{x^2+144}} + \frac{x-30}{\sqrt{x^2-60x+1684}} &= 0, \\
\frac{x}{\sqrt{x^2+144}} &= -\frac{x-30}{\sqrt{x^2-60x+1684}}, \\
x\sqrt{x^2-60x+1684} &= -(x-30)\sqrt{x^2+144}, \\
x^2(x^2-60x+1684) &= (x-30)^2(x^2+144), \\
x^4-60x^3+1684x^2 &= (900-60x+x^2)(x^2+144) \\
&= x^4-60x^3+1044x^2-8640x+129600, \\
640x^2+8640x-129600 &= 0, \\
320(x-9)(2x+45) &= 0, \\
\therefore x = -22.5,\ 9.
\end{aligned}$$

Hence, W has one critical number 9. Note that

x	$W(x)$	
0	53.04	
9	50	smallest
30	60.31	

Hence, by the Closed Interval Method, the function W attains the absolute minimum at $x = 9$. Thus, the wire should be staked at 9 feet from the 12-foot pole.

4. Four feet of wire is used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?
Answer:



Let x and r be the side length of the square and the radius of circle, respectively. Then the the sum of two areas from square and circle is $x^2 + \pi r^2$. Note that the length of wire used for the square is $4x$ and the remainder of the total wire is $4 - 4x$, which will be made into a circle. This gives a rise to the equation $2\pi r = 4 - 4x$. Hence, the objective function and the constraint equation are given by the following:

$$\begin{cases} A = x^2 + \pi r^2 & \text{objective function} \\ 2\pi r = 4x & \text{constraint equation} \end{cases}$$

From the constraint equation $2\pi r = 4x$, we can eliminate r in the objective function; by substituting $r = \frac{4-4x}{2\pi} = \frac{2-2x}{\pi}$, we get

$$A(x) = x^2 + \pi \left(\frac{2-2x}{\pi} \right)^2 .$$

Now we need to find out the domain of A. First, $x \geq 0$ since x is a side length of a square. Note that x can be zero because that will make all the wire used for the circle. Second, from the constraint equation $r = \frac{2-2x}{\pi}$, we know that $r \geq 0$ by the similar reasoning. So we get another inequality $x \leq 1$. Combining these two inequalities, we get $0 \leq x \leq 1$. Hence, we obtain the following absolute maximum problem:

find where the function $A(x) = x^2 + \pi \left(\frac{2-2x}{\pi} \right)^2$ on $[0, 1]$ attains the absolute minimum.

Note that $A'(x) = 2x - \frac{4(2-2x)}{\pi}$. Solving $A'(x) = 0$, we get $x = \frac{4}{\pi+4}$. Hence, A has one critical number $\frac{4}{\pi+4}$. Note that

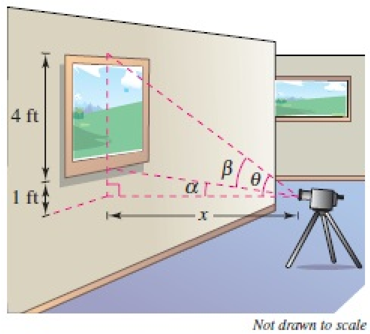
x	$A(x)$	
0	$\frac{4}{\pi} = 1.27$	largest
$\frac{4}{\pi+4}$	0.56	
1	1	

Hence, by the Closed Interval Method, the function A attains the absolute maximum at $x = 0$. This means that no wire should be use for the square, i.e., the entire 4 ft wire must be use to make a circle to enclose the maximum total area.

5. A photographer is taking a picture of a four-foot painting hung in an art gallery. The camera lense is 1 foot below the lower edge of the painting, as shown in the below figure. How far should the camera be from the painting to maximize the angle subtended by the camera lens?

(Use the trigonometric formula $\tan(\theta-\alpha)=\dfrac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}$.)

Answer:



We want to find x when the angle β is maximized. That is we need write β as a function of x and then find out where the function β attains its absolute maximum.
From the picture, we find that

$$\tan\alpha=\frac{1}{x},\quad \tan\theta=\frac{5}{x}.$$

By the given trig identity, we have

$$\begin{aligned}\tan\beta &= \tan(\theta-\alpha)\\ &= \frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}\\ &= \frac{\frac{5}{x}-\frac{1}{x}}{1+\frac{5}{x}\cdot\frac{1}{x}}\\ &= \frac{4x}{x^2+5}.\end{aligned}$$

Hence, by taking the arctangent both sides, we find that

$$\beta(x)=\arctan\left(\frac{4x}{x^2+5}\right).$$

Since x is the length, we must have $x>0$, which means that the domain of the function β is $(0,\infty)$. So we need to solve the following problem:

find where the function $\beta(x)$ on $(0,\infty)$ attains the absolute maximum.

Note that

$$\beta'(x)=\frac{1}{1+\left(\frac{4x}{x^2+5}\right)^2}\cdot\frac{4(x^2+5)-4x(2x)}{(x^2+5)^2}=\frac{-4(x^2-5)}{(x^2+5)^2+(4x)^2}=\frac{-4(x-\sqrt{5})(x+\sqrt{5})}{(x^2+5)^2+(4x)^2}.$$

Since the denominator of $\beta'(x)$ is always greater than zero when $x>0$, critical numbers can only be obtained from the solution of $\beta'(x)=0$. By solving the equation $\beta'(x)=0$, we get $x=-\sqrt{5}$ and $\sqrt{5}$. Hence, β has one critical number $\sqrt{5}$. We can check that $\beta'(x)$ changes its sign from positive to negative at $x=\sqrt{5}(\because\ \beta'(1)>0$ and $\beta'(3)<0.)$ Hence, β attains a local maximum at $x=\sqrt{5}$ by the First Derivative Test. Since β has only one critical number and β attains a local maximum there, β must attain the absolute maximum at $x=\sqrt{5}$. Thus, the camera must be placed $\sqrt{5}$ ft from the painting to maximize the angle subtended by the camera lens.