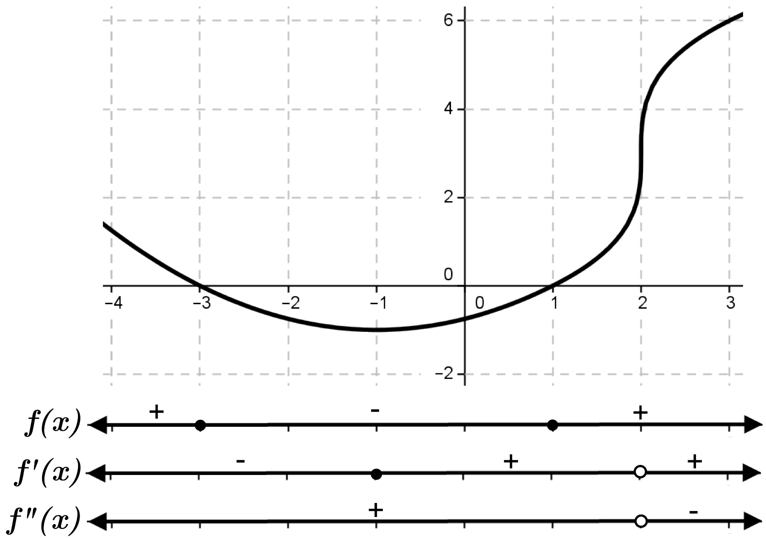
1. In the graph of $f(x)$ below, number lines are used to mark where $f(x)$ is zero/positive/negative/undefined, where $f'(x)$ is zero/positive/negative/undefined, and where $f''(x)$ is zero/positive/negative/undefined. Closed circles on the number lines indicate a zero-value, and open-circles indicate an undefined value.



   (a) What do the closed circles on the number line for $f(x)$ correspond to on the graph of $f(x)$?

   Solution: This is where $f(x)$ has x-intercepts (zeroes).

   (b) How does each $+$ or $-$ sign on the number line for $f(x)$ relate to the graph?

   Solution: The $+$ and $-$ signs indicate where $f(x)$ is positive or negative.

   (c) What does the closed circle at $x = -1$ on the number line for $f'(x)$ correspond to on the graph of $f(x)$?

   Solution: This is where $f(x)$ has a horizontal tangent line.

   (d) What does the open circle at $x = 2$ on the number line for $f'(x)$ correspond to on the graph of $f(x)$?

   Solution: This is where $f(x)$ has a vertical tangent line, so its slope, and $f'(x)$, is undefined here.

   (e) How does each $+$ or $-$ sign on the number line for $f'(x)$ relate to the graph?

   Solution: They indicate where $f'(x)$ is positive or negative. $f'(x)$ is negative where $f(x)$ is decreasing and positive where $f(x)$ is increasing.

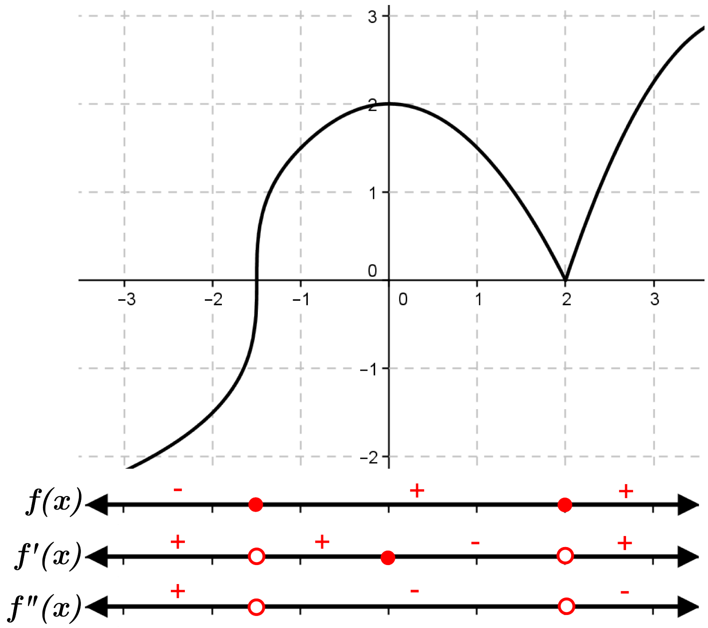
   (f) What does the open circle at $x = 2$ on the number line for $f''(x)$ correspond to on the graph of $f(x)$?

   Solution: Since $f'(x)$ is undefined here, so is $f''(x)$.

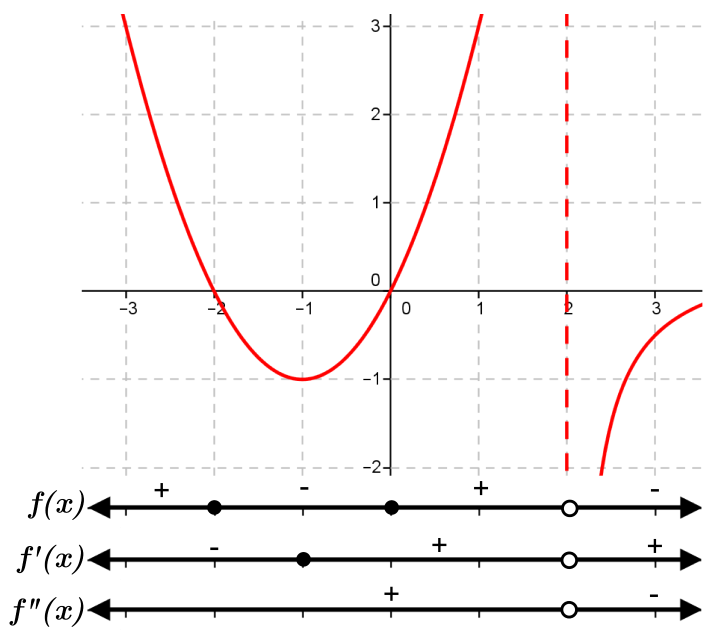
   (g) How does each $+$ or $-$ sign on the number line for $f''(x)$ relate to the graph?

   Solution: They indicate where $f''(x)$ is positive or negative. $f''(x)$ is negative where $f(x)$ is concave down and positive where $f(x)$ is concave up.

2. For the graph of $f(x)$ shown below, fill in the number lines for $f(x)$, $f'(x)$ and $f''(x)$, marking closed circles where there is a zero, marking open circles for undefined points, and marking $+$ and $-$ signs on each interval to show positive/negativeness.

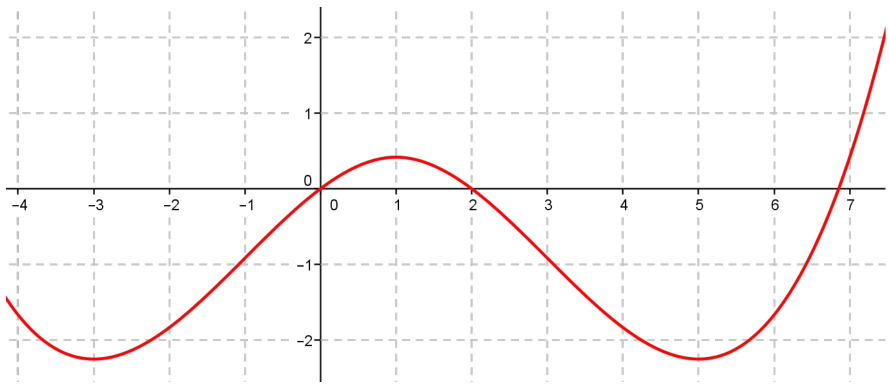


3. Draw a graph of $f(x)$ that fits the information shown in the number lines.

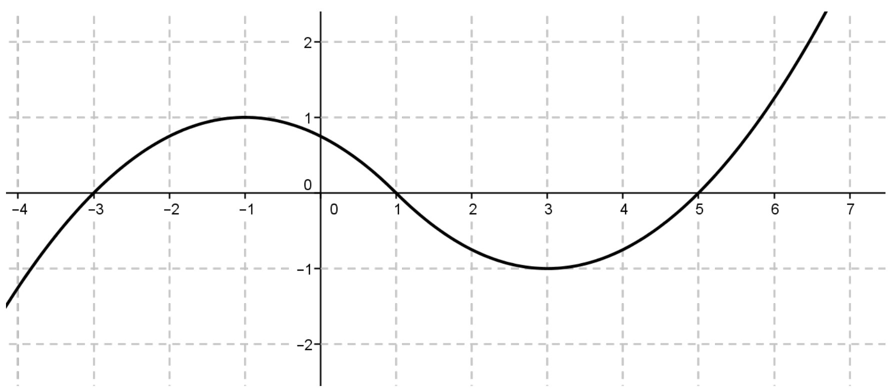


4. The middle graph drawn below shows $f'(x)$. Using the principles you learned in the previous problem, draw a possible graph of $f(x)$ above it, and a graph of $f''(x)$ below it. (If you are stuck try drawing the graph of $f''(x)$ first.)

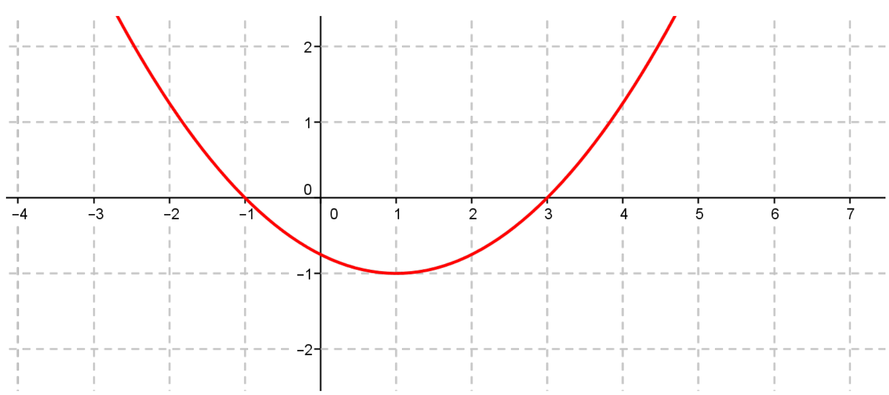
$f(x)$:



$f'(x)$:



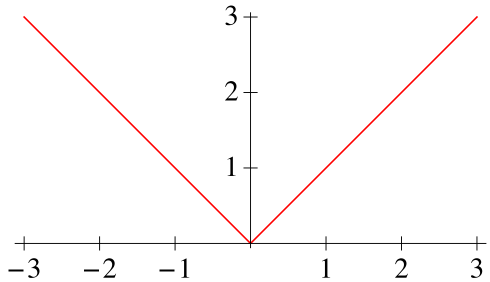
$f''(x)$:



5. This problem investigates the derivative of the absolute value function. Recall that we define the absolute value as:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

(a) In the space provided, draw a graph of the function $f(x) = |x|$.



(b) Using your graph from part (a), and your understanding of the derivative as the rate of change/slope of the tangent line, find the derivative function $f'(x)$ of the above function $f(x) = |x|$, for x not equal to 0 (fill in the blanks):

$$f'(x) = \begin{cases} \underline{\qquad\qquad} & \text{if } x > 0, \\ \underline{\qquad\qquad} & \text{if } x < 0. \end{cases}$$

Solution: 1 if $x > 0$

Solution: -1 if $x < 0$

.

(c) But what about $f'(0)$? It is not so clear from the picture even how to draw a tangent line to the function at the origin. So let's try to compute $f'(0)$ by first looking at the corresponding lefthand and righthand limits of the difference quotient.

i. Compute $\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$. Hint: use the piecewise definition of $f(x)$ given above.

Solution:

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|-|0|}{h} = \lim_{h\to 0^+} \frac{h-0}{h} = \lim_{h\to 0^+} 1 = 1$$

(since $h \to 0^+$ means h is positive, so $|h| = h$).

ii. Compute $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$.

Solution:

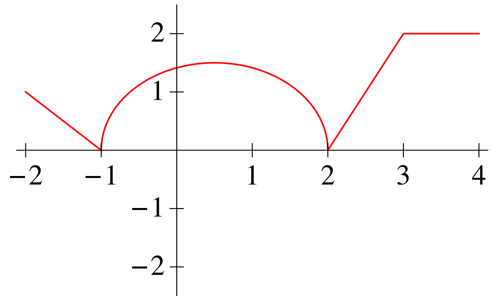
$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|-|0|}{h} = \lim_{h\to 0^-} \frac{-h-0}{h} = \lim_{h\to 0^-} -1 = -1$$

(since $h \to 0^-$ means h is negative, so $|h| = -h$).

iii. What do your answers to parts (i) and (ii) tell you about $f'(0)$? Please explain.

Solution: Since the righthand limit $\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$ does NOT equal the lefthand limit $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$, the (two-sided) limit $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist. But this (two-sided) limit is $f'(0)$, so $f'(0)$ does not exist.

6. Using what you've learned above, sketch the graph of a *continuous* function $g(x)$ such that $g(x)$ is not differentiable at $x = -1$, $x = 2$, nor $x = 3$.



7. Create a piecewise function where one piece is a quadratic function and the other piece is a linear function which is continuous everywhere but not differentiable at $x = 0$.

$$f(x) = \begin{cases} \underline{\hspace{3cm}} & \text{if } x \geq 0, \\ \underline{\hspace{3cm}} & \text{if } x < 0. \end{cases}$$

.

Solution:

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ x & \text{if } x < 0. \end{cases}$$

.

8. Create a piecewise function where one piece is a quadratic function and the other piece is a linear function which is continuous and differentiable everywhere.

$$f(x) = \begin{cases} \underline{\hspace{3cm}} & \text{if } x > 0, \\ \underline{\hspace{3cm}} & \text{if } x \leq 0. \end{cases}$$

.

Solution:

$$f(x) = \begin{cases} (x+1)^2 - 1 & \text{if } x \geq 0 \\ 2x & \text{if } x < 0. \end{cases}$$

.

Note, because $\left.\frac{d}{dx}\right|_{x=0} (x+1)^2 - 1 = 2(0+1) = 2$ and $\left.\frac{d}{dx}\right|_{x=0} 2x = 2$, so the right and left difference quotient limits agree at $x = 0$ and therefore the function is differentiable at $x = 0$.