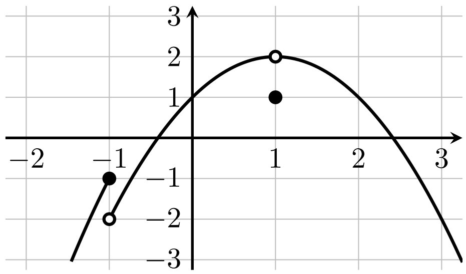
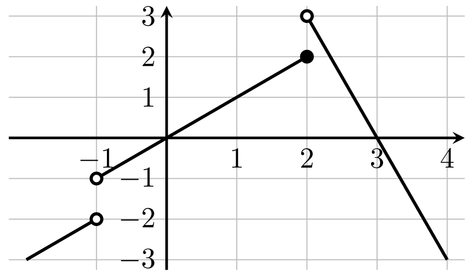
Graphical Limits Using Limit Laws

Graph of f



Graph of g



1. $\lim_{x\to 0}(f(x)+g(x))$

   Solution: $\lim_{x\to 0}(f(x)+g(x)) = \lim_{x\to 0} f(x) + \lim_{x\to 0} g(x) = 1+0 = 1$

2. $\lim_{x\to 1}(f(x)g(x))$

   Solution: $\lim_{x\to 1}(f(x)g(x)) = \lim_{x\to 1} f(x) \cdot \lim_{x\to 1} g(x) = 2 \cdot 1 = 2$

3. $\lim_{x\to 1}(f(x)+g(x))$

   Solution: $\lim_{x\to 1}(f(x)+g(x)) = \lim_{x\to 1} f(x) + \lim_{x\to 1} g(x) = 2+1 = 3$

4. $\lim_{x\to 2^+}(2f(x)+3g(x))$

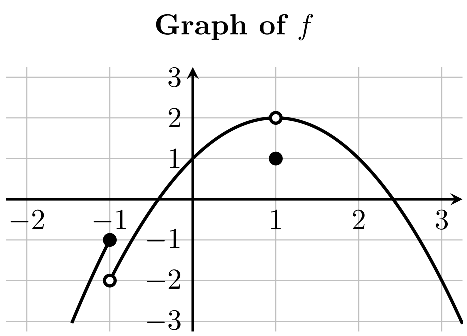
   Solution: $\lim_{x\to 2^+}(2f(x)+3g(x)) = 2\cdot \lim_{x\to 2^+} f(x) + 3\cdot \lim_{x\to 2^+} g(x) = 2\cdot 1 + 3\cdot 3 = 11$

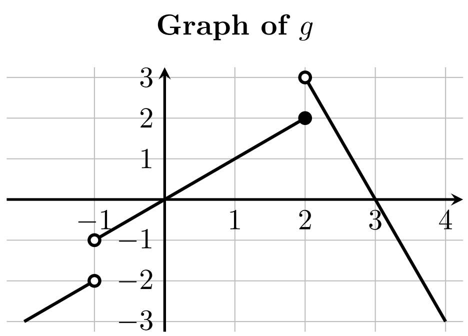
5. $\lim_{x\to 2^-}(x^2+(\ln x)\cdot g(x))$

   Solution: $\lim_{x\to 2^-}(x^2+(\ln x)\cdot g(x)) = \lim_{x\to 2^-} x^2 + \lim_{x\to 2^-}(\ln x \cdot g(x)) = 4 + \lim_{x\to 2^-} \ln x \cdot \lim_{x\to 2^-} g(x) =$ $4 + (\ln 2)\cdot 2 = 4 + \ln 4$

6. $\lim_{x\to 2}(f(x)-g(x))$

   Solution: First evaluate the two one-sided limits:
   $\lim_{x\to 2^+}(f(x)-g(x)) = \lim_{x\to 2^+} f(x) - \lim_{x\to 2^+} g(x) = 1-3 = -2$
   $\lim_{x\to 2^-}(f(x)-g(x)) = \lim_{x\to 2^-} f(x) - \lim_{x\to 2^-} g(x) = 1-2 = -1$
   We see that $\lim_{x\to 2^+}(f(x)-g(x)) \neq \lim_{x\to 2^-}(f(x)-g(x))$, so $\lim_{x\to 2}(f(x)-g(x))$ does not exist.





7. $\lim_{x\to 3} \frac{g(x)}{f(x)}$

Solution: $\lim_{x\to 3} \frac{g(x)}{f(x)} = \frac{\lim_{x\to 3} g(x)}{\lim_{x\to 3} f(x)} = \frac{0}{-2} = 0$

8. $\lim_{x\to 3^+} \frac{f(x)}{g(x)}$

Solution: We anticipate an issue because the limit of the denominator is 0, so we'll check the limits of the numerator and denominator separately. $\lim_{x\to 3^+} f(x) = -2$ and $\lim_{x\to 3^+} g(x) = 0$. The sign of the denominator is negative as x approaches 3 from the right. We have a non-zero limit divided by a number approaching 0 from below (which we can think of as the form "$\frac{-2}{0^-}$"), so $\lim_{x\to 3^+} \frac{f(x)}{g(x)} = +\infty$.

9. $\lim_{x\to 3} \frac{f(x)}{g(x)}$

Solution: From the previous problem, we know that we are dealing with a limit involving infinity, which tells us that we need to consider two one-sided limits. We already know that the limit from the right is $+\infty$, so next we'll look at the limit from the left. The limit of the numerator is $\lim_{x\to 3^-} f(x) = -2$ and the limit of the denominator is $\lim_{x\to 3^-} g(x) = 0$. This time the sign of the denominator is positive as x approaches 3 from the left. We have a non-zero limit divided by a number approaching 0 from below (which we can think of as the form "$\frac{-2}{0^+}$"), so $\lim_{x\to 3^-} \frac{f(x)}{g(x)} = -\infty$. Finally, $\lim_{x\to 3^+} \frac{f(x)}{g(x)} \neq \lim_{x\to 3-} \frac{f(x)}{g(x)}$, so $\lim_{x\to 3} \frac{f(x)}{g(x)}$ does not exist.

10. $\lim_{x\to 1} \sqrt{1 + f(x) + g(x)}$

Solution: $\lim_{x\to 1} \sqrt{1 + f(x) + g(x)} = \sqrt{\lim_{x\to 1}(1 + f(x) + g(x))} = \sqrt{1 + \lim_{x\to 1} f(x) + \lim_{x\to 1} g(x)} = \sqrt{1 + 2 + 1} = 2$

11. $\lim_{x\to -1} (f(x) + g(x))$

Solution: Since neither $\lim_{x\to -1} f(x)$ nor $\lim_{x\to -1} g(x)$ exists, we cannot use limit laws to break apart the limit. The jumps in both graphs at $x = -1$ hint to us to try two one-sided limits. Since these limits exist, we can then use the limit laws to break apart each of the one-sided limits.

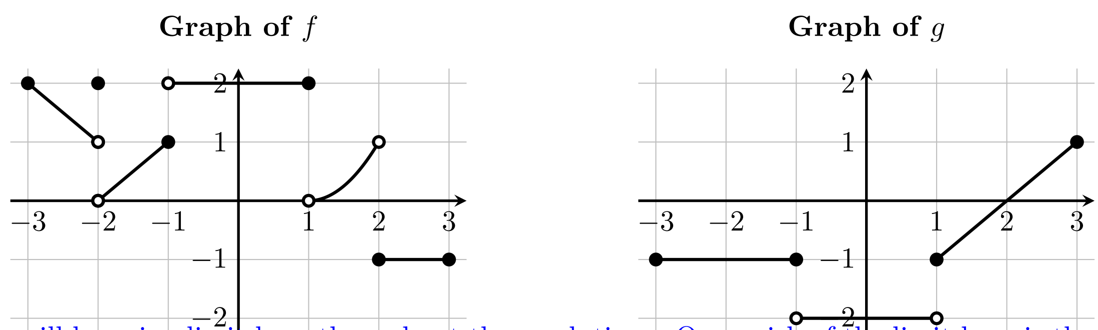
$$\lim_{x\to -1^+}(f(x)+g(x)) = \lim_{x\to -1^+} f(x) + \lim_{x\to -1^+} g(x) = -2 + -1 = -3$$

$$\lim_{x\to -1^-}(f(x)+g(x)) = \lim_{x\to -1^-} f(x) + \lim_{x\to -1^-} g(x) = -1 + -2 = -3$$

Since $\lim_{x\to -1^+}(f(x)+g(x)) = \lim_{x\to -1^-}(f(x)+g(x)) = -3$, we have $\lim_{x\to -1}(f(x)+g(x)) = -3$. This is a little surprising - the jump discontinuities of the two functions manage to cancel each other out, and $f(x)+g(x)$ does have a limit at $x = -1$.

Wacky Limits

Problem: These limits are wacky. Help me understand the key. All I have is the answers and not the reasons why the answers are what they are. Do this by providing the correct mathematical reasons/work explaining how one gets the correct answer.



We will be using limit laws throughout these solutions. One quirk of the limit laws is that they can only be applied if the individual limits exist. For example, the limit law equation

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

is only true if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and $\lim_{x\to a} g(x)$ is not equal to 0 (otherwise the fraction on the right-hand side doesn't make sense). Accordingly, when we use limit laws, we will use them provisionally. If all limits involved turn out to exist, the limit law will work, and we will have our answer. If they do not exist, the limit law equation will be nonsense and we will have to try something else.

1. $\lim_{x\to 0}(f(x)+g(x)) = 0$

   Solution: Applying the limit law for sums (provisionally),

   $$\lim_{x\to 0}(f(x)+g(x)) = \lim_{x\to 0} f(x) + \lim_{x\to 0} g(x).$$

   The value of $f(x)$ is 2 whenever x is near 0, so as x approaches 0, $f(x)$ approaches 2. In other words, $\lim_{x\to 0} f(x) = 2$. The value of $g(x)$ is -2 whenever x is near 0, so $\lim_{x\to 0} g(x) = -2$. Substituting into the equation from above,

   $$\lim_{x\to 0}(f(x)+g(x)) = 2 + (-2) = 0.$$

2. $\lim_{x\to 2^-}\frac{g(x)}{f(x)} = \lim_{x\to 2^+}\frac{g(x)}{f(x)} = \lim_{x\to 2}\frac{g(x)}{f(x)} = 0$

Solution: Let's consider $\lim_{x\to 2^-}\frac{g(x)}{f(x)}$ first. This is a left-hand limit, so we want to know the value that $\frac{g(x)}{f(x)}$ approaches when x approaches 2 from the left. Applying the limit law for fractions,

$$\lim_{x\to 2^-}\frac{g(x)}{f(x)}=\frac{\lim_{x\to 2^-}g(x)}{\lim_{x\to 2^-}f(x)}.$$

To find the value of the limit $\lim_{x\to 2^-}g(x)$, we pick an x-value a little less than 2 and see what happens to the value of $g(x)$ as x increases to 2. In this case, $g(x)$ approaches 0 as x approaches 2 from the left, so $\lim_{x\to 2^-}g(x)=0$. While it is true that 0 divided by any number (besides 0) is zero, we are still not done yet, since we need to confirm that the limit for $f(x)$ exists and is not zero (since in that case the limit law equation would not apply). To find the value of the limit $\lim_{x\to 2^-}f(x)$, we pick an x-value a little less than 2 and see what happens to the value of $f(x)$ as x increases to 2. In this case, $f(x)$ approaches 1 as x approaches 2 from the left, so $\lim_{x\to 2^-}f(x)=1$. Substituting into the equation from above,

$$\lim_{x\to 2^-}\frac{g(x)}{f(x)}=\frac{0}{1}=0.$$

Next, we consider the right-hand limit $\lim_{x\to 2^+}\frac{g(x)}{f(x)}$. This time,

$$\lim_{x\to 2^+}\frac{g(x)}{f(x)}=\frac{\lim_{x\to 2^+}g(x)}{\lim_{x\to 2^+}f(x)}.$$

From the graph, $\lim_{x\to 2^+}g(x)=0$ and $\lim_{x\to 2^+}f(x)=-1$, so

$$\lim_{x\to 2^+}\frac{g(x)}{f(x)}=\frac{0}{-1}=0.$$

Since both the left- and right-hand limits exist and agree, we have that the two-sided limit $\lim_{x\to 2}\frac{g(x)}{f(x)}$ is also 0, which is what we wanted to show.

3. $\lim_{x\to -1}(f(x)+g(x))=0$

Solution: Since $\lim_{x\to -1^-} f(x) = 1$ while $\lim_{x\to -1^+} f(x) = 2$, we know that $\lim_{x\to -1} f(x)$ does not exist. This means we cannot apply the limit laws for $\lim_{x\to -1} f(x)$. We fall back to our standard trick: taking left- and right-hand limits individually. Applying the limit law for sums to the left-hand limit,

$$\begin{aligned}\lim_{x\to -1^-}(f(x)+g(x)) &= \lim_{x\to -1^-} f(x) + \lim_{x\to -1^-} g(x)\\ &= 1 + (-1)\\ &= 0.\end{aligned}$$

Similarly, for the right-hand limit we get

$$\begin{aligned}\lim_{x\to -1^+}(f(x)+g(x)) &= \lim_{x\to -1^+} f(x) + \lim_{x\to -1^+} g(x)\\ &= 2 + (-2)\\ &= 0.\end{aligned}$$

Since the two one-sided limits exist and agree, the two-sided limit $\lim_{x\to -1}(f(x)+g(x))$ exists and is also equal to 0, which is what we wanted to show.

4. $\lim_{x\to -1} \frac{f(x)}{g(x)} = -1$

Solution: Since $\lim_{x\to -1^-} f(x) = 1$ while $\lim_{x\to -1^+} f(x) = 2$, we know that $\lim_{x\to -1} f(x)$ does not exist. This means that we cannot apply the limit laws for $\lim_{x\to -1} f(x)$. This forces us to fall back to our standard trick: taking left- and right-hand limits individually. Applying the limit law for quotients to the left-hand limit,

$$\begin{aligned}\lim_{x\to -1^-}\frac{f(x)}{g(x)} &= \frac{\lim_{x\to -1^-} f(x)}{\lim_{x\to -1^-} g(x)}\\ &= \frac{1}{-1}\\ &= -1.\end{aligned}$$

Similarly, for the right-hand limit we get

$$\begin{aligned}\lim_{x\to -1^+}\frac{f(x)}{g(x)} &= \frac{\lim_{x\to -1^+} f(x)}{\lim_{x\to -1^+} g(x)}\\ &= \frac{2}{-2}\\ &= -1.\end{aligned}$$

Since the two one-sided limits exist and agree, the two-sided limit $\lim_{x\to -1} \frac{f(x)}{g(x)}$ exists and is equal to -1, which is what we wanted to show.

5. $\lim_{x\to 2}(f(x)g(x)) = 0$

Solution: Applying the limit law for products (provisionally),

$$\lim_{x\to 2}(f(x)g(x)) = \left(\lim_{x\to 2} f(x)\right) \cdot \left(\lim_{x\to 2} g(x)\right)$$

Now from the graph, $\lim_{x\to 2} g(x) = 0$. It is true that any number times 0 is 0, but we are not done yet, since we still need to check that $\lim_{x\to 2} f(x)$ exists. Unfortunately, it does not, since $\lim_{x\to 2^-} f(x) = 1$ while $\lim_{x\to 2^+} f(x) = -1$. This means we cannot use the limit law equation above, so we will have to try something else. We fall back to our standard trick of considering two one-sided limits. Applying the limit law for products to the left-hand limit,

$$\begin{aligned}\lim_{x\to 2^-}(f(x)g(x)) &= \left(\lim_{x\to 2^-} f(x)\right) \cdot \left(\lim_{x\to 2^-} g(x)\right)\\ &= 1 \cdot 0\\ &= 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{x\to 2^+}(f(x)g(x)) &= \left(\lim_{x\to 2^+} f(x)\right) \cdot \left(\lim_{x\to 2^+} g(x)\right)\\ &= -1 \cdot 0\\ &= 0.\end{aligned}$$

Since the two one-sided limits exist and agree, the two-sided limit $\lim_{x\to 2}(f(x)g(x))$ exists and is equal to 0, which is what we wanted to show.

6. $\lim_{x\to 3^-} f(g(x)) = 2$

Solution: We need to visualize what happens to $f(g(x))$ when x approaches 3 from the left. Since we find $f(g(x))$ by plugging $g(x)$ into f, a good way to start is to think of how $g(x)$ behaves when x approaches 3 from the left.
From the graph, we notice that $\lim_{x\to 3^-} g(x) = 1$. Moreover, $g(x)$ *increases* to 1 as x increases to 3. This means

$$\lim_{x\to 3^-} f(g(x)) = \lim_{x\to 1^-} f(x).$$

We are taking the limit as x approaches 1 from the left since $g(x)$ approaches 1 from the left. Now, from the graph, $\lim_{x\to 1^-} f(x) = 2$, so $\lim_{x\to 3^-} f(g(x)) = 2$.

7. $\lim_{x\to 1^+} f(g(x)) = 2$

Solution: From the graph, $\lim_{x\to 1^+} g(x) = -1$. Moreover, $g(x)$ *decreases* to -1 as x decreases to 1. This means

$$\lim_{x\to 1^+} f(g(x)) = \lim_{x\to -1^+} f(x).$$

We are approaching -1 from the right since $g(x)$ approaches -1 from the right. From the graph, $\lim_{x\to -1^+} f(x) = 2$, so $\lim_{x\to 1^+} f(g(x)) = 2$.

8. $\lim_{x\to -2^-} g(f(x)) = -1$ (and NOT -2)

Solution: From the graph, $\lim_{x\to-2^-} f(x) = 1$. Moreover, $f(x)$ is *decreasing* to 1 as x increases to -1. This means

$$\lim_{x\to-2^-} g(f(x)) = \lim_{x\to 1^+} g(x).$$

We are approaching 1 from the right since $f(x)$ approaches 1 from the right. From the graph, $\lim_{x\to1^+} g(x) = -1$, so $\lim_{x\to-2^-} g(f(x)) = -1$.

9. $\lim_{x\to1^-} f(g(x)) = 2$ (and NOT 1)

Solution: To get a feel for the problem, we start with a table of values:

x	$g(x)$	$f(g(x))$
0	-2	2
0.9	-2	2
0.99	-2	2
0.999	-2	2

Notice $g(x)$ is exactly -2 whenever x is sufficiently close to 1 and less than 1. This means that $f(g(x))$ is $f(-2)$ for these same x-values. Therefore

$$\lim_{x\to1^-} f(g(x)) = f(-2) = 2.$$

10. $\lim_{x\to2^-} \dfrac{f(x)}{g(x)} = -\infty$

Solution: We start by applying the limit law for quotients (provisionally):

$$\lim_{x\to2^-} \frac{f(x)}{g(x)} = \frac{\lim_{x\to2^-} f(x)}{\lim_{x\to2^-} g(x)}$$

However, $\lim_{x\to2^-} g(x) = 0$, so the limit law does not apply. We are forced to reason about the limit another way.

So we consider the values of the fraction $\dfrac{f(x)}{g(x)}$ when x is near 2 and less than 2. Since $\lim_{x\to2^-} f(x) = 1$, $f(x)$ should be close to 1 for these x values. On the other hand, $g(x)$ is negative and approaching 0 for these x values. That is, for x near 2 and less than 2,

$$\frac{f(x)}{g(x)} \approx \frac{1}{\text{small negative number}}$$

which is a large negative number. Since the small negative number can be made as small as we like by choosing x close enough to 2, the fraction $\dfrac{f(x)}{g(x)}$ can be made as large a negative number as we like. That is, $\lim_{x\to2^-} \dfrac{f(x)}{g(x)} = -\infty$.

11. $\lim_{x\to2^+} \dfrac{f(x)}{g(x)} = -\infty$.

Solution: We consider the values of the fraction $\frac{f(x)}{g(x)}$ when x is near 2 and greater than 2. Since $\lim_{x\to 2^+} f(x) = -1$, $f(x)$ is near -1 for these x values. On the other hand, $g(x)$ is positive and approaching 0 for these x values. That is, for x near 2 and greater than 2,

$$\frac{f(x)}{g(x)} \approx \frac{-1}{\text{small postive number}}$$

which is a large negative number. Since the small positive number can be made as small as we like by choosing x close enough to 2, the fraction $\frac{f(x)}{g(x)}$ can be made as large a negative number as we like. That is, $\lim_{x\to 2^+} \frac{f(x)}{g(x)} = -\infty$.

12. $\lim_{x\to 2} \frac{f(x)}{g(x)} = -\infty$

Solution: From the previous two problems, the two one-sided limits both tend to $-\infty$, so the two-sided limit $\lim_{x\to 2} \frac{f(x)}{g(x)}$ also approaches $-\infty$.