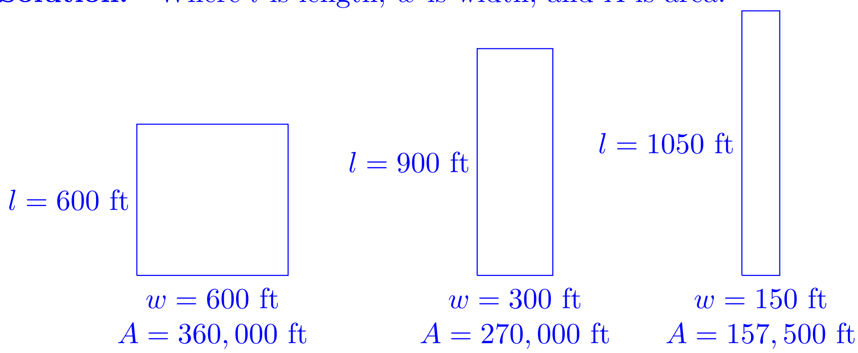
1. A farmer has 2400 feet of fencing and wants to use it to fence off a rectangular field. What are the dimensions of the field that has the largest area, and what is that largest area?
   The goal is to model this situation with a function (like we did in Project 1), then use the techniques of Chapter 4 to find the absolute maximum.

Step 1: Draw a picture of several possible fields. Label the pictures by assigning variables to any quantities that change. List any other variables that might be important.

Solution: Where l is length, w is width, and A is area.



Step 2: Which quantity from the previous part is the one that we want to maximize?

Solution: We need to maximize area, the A variable.

Step 3: Use basic geometry to write a formula for the variable you named in the previous part. If you end up with a function that has two independent variables (input variables), that's a problem we will have to fix in the next step.

Solution: $A = l \times w$

Step 4: Turn the constraint that we have only 2400 feet of fencing into an equation. Then use this equation to eliminate one of the variables in Step 3. You now have a function of one independent variable (input variable), and this is the function to maximize.

Solution: Perimeter $= 2 \cdot l + 2 \cdot w$, so we have $2400 = 2 \cdot l + 2 \cdot w$. Solving for l, we have

$$l = 1200 - w.$$

We now plug this into our equation for A to get

$$A = (1200 - w)w.$$

This is the equation we want to maximize.

Step 5: What is the domain? (Step 6 will be easier if you actually allow the possibility of "silly" rectangles with no area).

Solution: We only have 2400 feet of fencing to work with, so we must have $w+l \leq 1200$. It only makes sense for $l \geq 0$, and w is biggest when l is smallest. So taking $l = 0$, we see that $w \leq 1200$. The domain is $[0, 1200]$. We include the endpoints because it is easier to find a maximum on a closed interval, and A is defined at the endpoints.

Step 6: Use one of the procedures you know to find the absolute maximum value on the domain.

Solution: We first find $\frac{dA}{dw} = 1200 - 2w$. We set this equal to zero and solve for w to find the critical point $w = 600$. Finally, we evaluate A for $w = 0$, $w = 600$, and $w = 1200$. $A(0) = 0$ ft^2, $A(600) = 360,000$ ft^2, and $A(1200) = 0$ ft^2. Therefore, A has a maximum value of $360,000$ ft^2 when $w = 600$ ft.

Step 7: Answer the questions asked: what are the dimensions of the field that has the largest area, and what is the largest area?

Solution: Putting everything together, we see that if $w = 600$ ft then $l = 600$ ft. Therefore, the dimensions which maximize the area of a field with a perimeter of 2400 ft are 600 ft $\times$ 600 ft field with a maximum area of $360,000$ ft^2.

2. A farmer has 2400 feet of fencing and this time wants to fence off a rectangular field that borders a straight river. The farmer needs no fence along the river. What are the dimensions of the field that has the largest area, and what is that largest area? (This problem is similar to problem 2; use the same sequence of steps in your solution.) Explain why your answer is different from Problem 1.

Solution: We need to maximize area, so we'll maximize $A = l \times w$. We are given the constraint $2400 = 2l + w$ (Note: we only need one width since the other is covered by the river).

We need to use the constraint to eliminate one of the right hand side variables. Solving the constraint for w we have $w = 2400 - 2l$. Substituting this into the area equation we have $A = l(2400 - 2l)$.

We only have 2400 feet of fencing to work with, so we must have $2l + w \leq 2400$. It only makes sense for $w \geq 0$, so taking $w = 0$, we see that $l \leq 1200$. Since it doesn't make sense to have negative lengths of fencing, the domain is $[0, 1200]$. Notice again we are including the "silly" rectangles corresponding to the endpoints of the interval to make the math simpler.

Now we can maximize $A = l(2400 - 2l)$ by solving $\frac{dA}{dl} = 0$ for l and comparing the critical points to the end points of the domain. $\frac{dA}{dl} = 2400 - 4l$; solving $0 = 2400 - 4l$ gives us the critical point $l = 600$.

Finally, we evaluate A for $l = 0$, $l = 600$, and $l = 1200$. $A(0) = 0$ ft^2, $A(600) = 720,000$ ft^2, and $A(1200) = 0$ ft^2. Therefore, A has a maximum value of $720,000$ ft^2 when $l = 600$ ft and $w = 1200$ ft.

The field with maximum area is different than in the previous problem, since we had to fence only one of the horizontal segments. It makes sense that the field would be longer in this direction since it requires less fencing. Notice that the total amount of fencing used in the vertical direction is the same as the amount used in the horizontal direction.

3. A square-bottomed box with no top has a fixed volume of 500 cm^3 (1/2 Liter). What is the minimum surface area?

   Solution: We need to minimize surface area. The box will have a bottom with a surface area of w^2 and 4 sides with surface area $l \times w$ so we'll minimize $SA = w^2 + 4l \times w$.

   Our constraint is $500 = l \times w^2$. Solving for l we have $l = 500w^{-2}$. Substituting this into the surface area function gives us $SA = w^2 + 4 \times 500w^{-2} \times w = w^2 + 2000w^{-1}$.

   Domain: Both l and w must be non-negative. We also check the limitations that the constraint might force on the domain. $500 = l \times w^2$, so $l = \frac{500}{w}$. This means that l cannot be 0. So the domain is l in the interval $(0, \infty)$. Note that in this case $w = 0$ cannot be added to the domain, because SA not defined there.

   Since we are looking for a absolute minimum on an open interval, we need to hope we only have one critical point and that it is a local min. Luckily, setting $\frac{dSA}{dw} = 2w - 2000w^{-2} = 0$ and solving for w gives us only one the critical point $w = 10$. Now $\frac{d^2SA}{dw^2}(10) = 6 > 0$ tells us our critical point is a local minimum by the second derivative test. Thus it is also a absolute minimum on the domain.Therefore a square-bottomed box with no top has a minimum surface area of $SA(10) = 100 + 2000/10 = 300\ \text{cm}^2$.

4. As in the previous problem, a square-bottomed box with no top has a fixed volume of 500 cm^3 (1/2 Liter). But this time the material for the bottom costs \$2 per cm^2 while the sides cost \$1 per cm^2. What dimensions give the minimum cost?

   Solution: This time we need to minimize cost. The box will have a bottom with a cost of $2w^2$ and 4 sides with a cost of $l \times w$ so we'll minimize $C = 2w^2 + 4l \times w$.

   Our constraint is $500 = l \times w^2$. Solving for l we have $l = 500w^{-2}$. Substituting this into the cost function gives us $C = 2w^2 + 4 \times 500w^{-2} \times w = 2w^2 + 2000w^{-1}$.

   Domain: Both l and w must be non-negative. We also check the limitations that the constraint might force on the domain. $500 = l \times w^2$, so $l = \frac{500}{w}$. This means that l cannot be 0. So the domain is l in the interval $(0, \infty)$.

   Since we are looking for a absolute minimum on an open interval, we need to hope we only have one critical point and that it is a local min. Luckily, setting $\frac{dC}{dw} = 4w - 2000w^{-2} = 0$ and solving for w gives us only one the critical point $w = \sqrt[3]{500}$. Now $\frac{d^2C}{dw^2}(\sqrt[3]{500}) > 0$ tells us our critical point is a local minimum by the second derivative test. Thus it is also a absolute minimum on the domain. Therefore the minimum cost occurs when $w = \sqrt[3]{500}$ and $l = \sqrt[3]{500}$.