## Area accumulation functions and the FTC: an analytical perspective

1. Let $F(x) = \int_3^x e^{5t}\,dt$

   (a) Find a formula for $F(x)$ by anti-differentiating and substituting.
   $F(x) = \int_3^x e^{5t}\,dt = \frac{1}{5}e^{5t}\Big|_3^x = \frac{1}{5}e^{5x} - \frac{1}{5}e^{15}.$

   (b) Differentiate to find $F'(x)$.
   Taking the derivative of the previous result with respect to $x$ gives $F'(x) = e^{5x}$.

   (c) Explain your result.
   Using the Evaluation Theorem (a part of the FTC), we found a formula for $F(x)$ by antidifferentiating, substituting and subtracting. Then when we found the derivative of $F'(x)$ we basically undid our work. So we ended up with the function inside the integral, with the upper limit of integration $x$ substituted for $t$.

   (d) Why does the lower limit of integration not affect the derivative?
   The lower limit is constant and only shifts the area accumulation function; it does not affect its rate of change. Another way to look at it is that when we took the derivative in part (b), that part of the function was a constant so its rate of change was 0.

   (e) Using what you noticed and learned above, find $\frac{d}{dx}\left[\int_{-5}^x \arctan t\,dt\right]$.
   $\arctan x$

In summary:

> The Fundamental theorem of Calculus, Part 1: If $f$ is continuous on $[a, b]$, then
> $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = \underline{\quad f(x) \quad}$ (for $a < x < b$).
> Worded differently, if $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = \underline{\quad f(x) \quad}$.

2. Let $F(x) = \int_4^{x^2} \cos t\,dt$

   (a) Find a formula for $F(x)$ by anti-differentiating.
   $F(x) = \int_4^{x^2} \cos t\,dt = \sin t\Big|_4^{x^2} = \sin\left(x^2\right) - \sin(4)$

   (b) Differentiate to find $F'(x)$. Look at your answer and notice how it relates to the definition of $F(x)$.
   $F'(x) = 2x\cos\left(x^2\right)$

(c) Using what you noticed and learned above, find $\dfrac{d}{dx}\left[\int_2^{\sin x} \ln t\, dt\right]$.

$\cos(x)\ln(\sin x)$

In summary:

If $F(x) = \int_0^{g(x)} f(t)\, dt$, then $F'(x) = \underline{\ f(g(x)) \cdot g'(x)\ }$.

3. If $F(x) = \int_x^0 f(t)\, dt$, what is $F'(x)$? Hint: notice that $F(x) = -\int_0^x f(t)\, dt$.

$F'(x) = -f(x)$

4. If $F(x) = \int_{3x}^{x^2} \sin t\, dt$, what is $F'(x)$? Hint: the integral can be broken into two parts, so $F(x) = \int_{3x}^{0} \sin t\, dt + \int_0^{x^2} \sin t\, dt$.

$F'(x) = 2x\sin x^2 - 3\sin 3x$

In summary:

If $F(x) = \int_{a(x)}^{b(x)} f(t)\, dt$, then $F'(x) = \underline{\ f(b(x)) \cdot b'(x) - f(a(x)) \cdot a'(x)\ }$

5. Use the above result to answer the following: if $F(x) = \int_{x^3}^{1-x} \dfrac{t+1}{t-1}\, dt$, what is $F'(x)$?

$$F'(x) = -\frac{1-x+1}{1-x-1} - 3x^2 \cdot \frac{x^3+1}{x^3-1} = \frac{2-x}{x} - 3x^2 \cdot \frac{x^3+1}{x^3-1}$$