QUANTUM GROUPOIDS

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WHAT

IS A QUANTUM GROUPOID?

FROM GROUPS TO QUANTUM GROUPS

CLASSICAL		QUANTUM
a semigroup is a space Γ with a map $\Gamma \times \Gamma \xrightarrow{m} \Gamma$ s.t.		a bialgebra is an algebra A with a morphism $A \xrightarrow{\Delta} A \otimes A$ s.t.

$$\begin{tikzcd} \Gamma \times \Gamma \times \Gamma \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times m"'] \arrow[dr, phantom, "\circlearrowright"] & \Gamma \times \Gamma \arrow[d, "m"] \\ \Gamma \times \Gamma \arrow[r, "m"'] & \Gamma \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "\mathrm{id} \otimes \Delta"] \\ A \otimes A \arrow[r, "\Delta \otimes \mathrm{id}"'] & A \otimes A \otimes A \end{tikzcd}$$

a finite group is a semigroup with a unit and inversion map	$\rightsquigarrow$	a f.d. Hopf algebra is a bialgebra with a counit and antipode
a locally compact group is a group with a suitable topology or measure (Weil)	$\rightsquigarrow$	a locally compact quantum group is a C^*-/W^*-bialgebra a with left/right Haar weights

AXIOMATICS

GROUPOID	QUANTUM GROUPOID

basic ingredients

• a base space G^0	• a base algebra B
• a total space G	• a total algebra A
• a target map $G \xrightarrow{t} G^0$	• a target morphism $B \xrightarrow{\alpha} A$
• a source map $G \xrightarrow{s} G^0$	• a source morphism $B^{\mathrm{op}} \xrightarrow{\beta} A$
• a multiplication $G_s{\times}_t G \xrightarrow{m} G$	• a comultiplication $A \to A_\beta{\times}_\alpha A$
	(if $A \circlearrowright H$, then $A_\beta{\times}_\alpha A \circlearrowright H_\beta{\otimes}_\alpha H$)

basic assumptions

• associativity of m	• coassociativity of Δ
• $t(\gamma\gamma') = t(\gamma)$	• $\Delta(\alpha(b)) = \alpha(b) \otimes 1$
• $s(\gamma\gamma') = s(\gamma')$	• $\Delta(\beta(b^{\mathrm{op}})) = 1 \otimes \beta(b^{\mathrm{op}})$
	• $[\alpha(B), \beta(B^{\mathrm{op}})] = 0$

Associated to a finite groupoid G, we have two quantum groupoids:

The Function Algebra

base algebra and total algebra:	$C(G^0)$ and $C(G)$,	where $\delta_g \delta_{g'}$ is $\delta_{g,g'} \delta_g$
target map and source map:	$C(G^0) \rightrightarrows C(G)$,	pull-back along t or s
comultiplication:	$C(G) \to C(G^{(2)})$,	$\delta_\gamma \mapsto \sum_{\gamma = \gamma' \gamma''} \delta_{\gamma'} \otimes \delta_{\gamma''}$

The Groupoid Algebra

base algebra and total algebra:	$\mathbb{C}G^0$ and $\mathbb{C}G$,	where $g \cdot g'$ is gg' or 0
target map and source map:	$\mathbb{C}G^0 \hookrightarrow \mathbb{C}G$,	the natural inclusion
comultiplication:	$\mathbb{C}G \to \mathbb{C}(G * G)$,	$g \mapsto g \otimes g$

($G * G$: all pairs (g, g') with same source, same target)

WHY

STUDY QUANTUM GROUPOIDS?

VARIANTS OF QUANTUM GROUPOIDS AND WHERE APPEARED

finite quantum groupoids (Nikshych & Vainerman, Böhm, ...)	$\leftrightsquigarrow$	invariants of 3-manifolds (Turaev)
partial compact quantum groups (De Commer & T.)	$\leftrightsquigarrow$	Tannaka-Krein duals of partial fusion C^*-categories
dynamical quantum groups (Etingof & Varchenko, Koelink & Rosengren, ...)	$\leftrightsquigarrow$	dynamical Yang-Baxter equation from physics
measured quantum groupoids (Enock & Lesieur & Vallin)	$\leftrightsquigarrow$	quantum symmetries of inclusions of II_1 factors
algebraic quantum groupoids (Lu, Xu, Böhm & Szlachányi, T. & Van Daele)	$\leftrightsquigarrow$	Pontrjagin duality for (quantum) groupoids

Work in progress: C^*-algebraic theory of locally compact quantum groupoids

How

ABOUT EXAMPLES?

A QUANTUM GROUPOID FROM CLASSICAL DATA

- For every space X, the full equivalence relation $X \times X$ is a groupoid:

$$\begin{tikzcd} x & y \arrow[l, "{(x,y)}"'] & z \arrow[l, "{(y,z)}"'] \arrow[ll, dashed, bend left, "{(x,z)}"] \end{tikzcd}$$

- If Γ acts *freely* on X, then $(X \times X)/_\Gamma$ is a groupoid over $X/_\Gamma$.

- For *any* action of a group Γ on X, we get a quantum groupoid

$$C_0(X) \rightrightarrows C_0(X) \rtimes \Gamma \ltimes C_0(X)$$

with comultiplication $f \rtimes \gamma \ltimes f' \mapsto (f \rtimes \gamma \ltimes 1) \otimes (1 \rtimes \gamma \ltimes f')$

- Roughly, if a quantum group Γ acts on an algebra B, we get a quantum groupoid $B \rtimes \Gamma \ltimes B^{\mathrm{op}}$ with base B.

QUANTUM TRANSFORMATION GROUPOIDS

- If a group Γ acts on a space X, we get a transformation groupoid $X \rtimes \Gamma$

$$\begin{tikzcd} x & x\gamma = x' \arrow[l, "\gamma"'] & x'\gamma' = x\gamma\gamma' \arrow[l, "\gamma'"'] \arrow[ll, dashed, bend left, "\gamma\gamma'"] \end{tikzcd}$$

with groupoid algebra $C_0(X) \rtimes \Gamma$ and function algebra $C_0(X) \otimes C_0(\Gamma)$

- If Γ acts on an algebra B, we get crossed product with canonical maps

$$B \to B \rtimes \Gamma \to (B \rtimes \Gamma) \otimes \mathbb{C}\Gamma$$

To obtain a quantum transformation groupoid, we also need a map

$$B^{\mathrm{op}} \to B \rtimes \Gamma, \qquad b^{\mathrm{op}} \mapsto \textstyle\sum_\gamma b_\gamma \rtimes \gamma$$

whose image commutes with $b' \rtimes e$ for all $b' \in B$, i.e., $b' b_\gamma = b_\gamma \gamma(b')$.

- Roughly, if Γ is a quantum group and B a braided-commutative Yetter-Drinfeld algebra, we obtain quantum transformation groupoids

$$B \rtimes \Gamma \quad \text{and} \quad B \rtimes \hat{\Gamma}.$$

A DEFORMATION OF $S^2 \rtimes \mathrm{SU}(2)$

- An important compact quantum group is $\mathrm{SU}_q(2)$, where $q \in (0,1]$:
  - $C(\mathrm{SU}_q(2)) = C^*\left(\alpha, \gamma : \text{ the matrix } u := \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \text{ is unitary}\right)$
  - $\Delta\colon C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2)) \otimes C(\mathrm{SU}_q(2))$ given by $u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$
- We have an inclusion $\mathbb{T} \hookrightarrow \mathrm{SU}_q(2)$ in the form of a $*$-homomorphism
$$C(\mathrm{SU}_q(2)) \xrightarrow{\pi} C(\mathbb{T}) \text{ given by } \alpha \mapsto z \text{ and } \gamma \mapsto 0,$$
and obtain a quantum homogeneous space $S^2_q = {}_{\mathbb{T}}\backslash \mathrm{SU}_q(2)$ in form of
$$C(S^2_q) = \{f \in C(\mathrm{SU}_q(2)) : (\pi \otimes \mathrm{id})\Delta(f) = 1 \otimes f\},$$
which is a braided-commutative Yetter-Drinfeld algebra for $\mathrm{SU}_q(2)$.
- We get a measured quantum groupoid $\mathcal{G} = L^\infty(S^2_q) \rtimes \mathrm{SU}_q(2)$, and
$$\mathcal{G} = L^\infty\left({}_{\mathbb{T}}\backslash \mathrm{SU}_q(2)\right) \rtimes \mathrm{SU}_q(2) \sim_M \mathbb{T} \ltimes L^\infty\left(\mathrm{SU}_q(2)/_{\mathrm{SU}_q(2)}\right) = L\mathbb{T}.$$

UNIVERSAL SYMMETRIES

- A f.d. algebra D has a quantum automorphism group $\mathrm{QAut}(D) = A$ with an action, that is, a homomorphism $D \xrightarrow{\delta} D \otimes A$ such that
$$\begin{tikzcd} D \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\circlearrowleft"] & D \otimes A \arrow[d, "\mathrm{id} \otimes \Delta"] \\ D \otimes A \arrow[r, "\delta \otimes \mathrm{id}"'] & D \otimes A \otimes A \end{tikzcd}$$
that is universal (every quantum group action on D is a quotient).
- For example, $\mathrm{QAut}(\mathbb{C}^n)$ is called a quantum permutation group.
- For every map $E \to X$, we have an automorphism groupoid
$$\mathrm{Aut}(E \to X) = \coprod_{x,y \in X} \mathrm{Iso}(E_x, E_y).$$
- To an inclusion of f.d. algebras $B \hookrightarrow D$, we can associate a quantum automorphism groupoid $\mathrm{QAut}(B \hookrightarrow D)$ with a universal action on D.