Mapping Spaces for Orbispaces

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Based on:

- Vesta Coufal, Dorette Pronk, Carmen Rovi, Laura Scull, Courtney Thatcher, Orbispaces and their mapping spaces via groupoids: a categorical approach, *Contemporary Mathematics* **641** (2015), pp. 135–166
- Dorette Pronk, Laura Scull, The Structure of Mapping Spaces for Orbispaces, in progress.

Other references:

- W. Chen, On a notion of maps between orbifolds, I. Function spaces, *Communications in Contemporary Mathematics* **8** (2006), pp. 569–620.

Today's Question

- For two orbifolds $\mathcal{G}$ and $\mathcal{H}$, can we define mapping object $Map(\mathcal{G}, \mathcal{H})$ which is itself an orbifold?

Outline

Orbifolds

- [Satake, 1956] Orbifolds (V-manifolds) defined as spaces with ‘mild’ singularities
- Given by an underlying space with an atlas of charts and embeddings.
- ‘Mild singularity’: quotient of Euclidean space by the action of a finite group.

Topological Groupoids

- A **topological groupoid** has a space of object $\mathcal{G}_0$ and a space of arrows $\mathcal{G}_1$, where all structure maps are continuous
- $\mathcal{G}$ is **étale** when s (and hence t) is a local homeomorphism
- $\mathcal{G}$ is **proper** when the diagonal,

$$(s,t)\colon \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0,$$

is a proper map (i.e., closed with compact fibers).

Orbigroupoids

Definition

- A topological groupoid is an **orbigroupoid** if it is both étale and proper.
- All isotropy groups are finite.
- The quotient space,

$$\mathcal{G}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{G}_0 \twoheadrightarrow X_{\mathcal{G}}$$

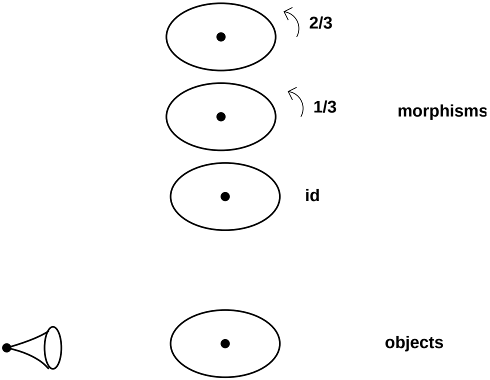
  is also called the **underlying space** of the orbigroupoid.
- This space is an orbifold.

Examples

A G-point

Examples

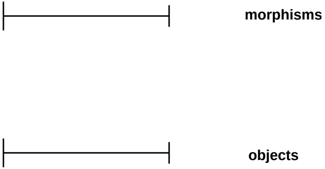
A Cone of Order 3



This is a translation groupoid, $\mathbb{Z}/3 \ltimes D$.

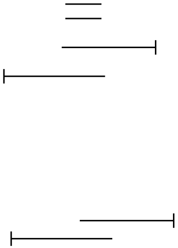
Examples

The Unit Interval



Examples

A Split Unit Interval

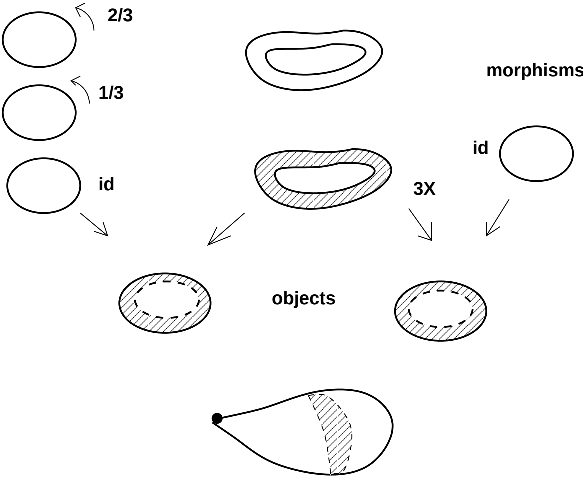


morphisms

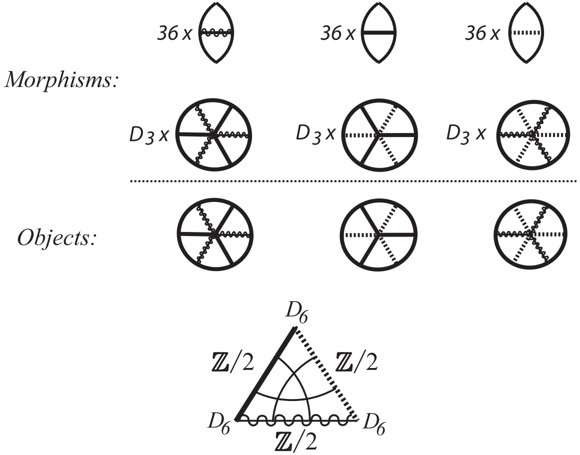
objects

Examples

The Teardrop Groupoid



Examples: The Triangular Billiard Groupoid $\mathbb{T}$



Maps

We can define maps between topological groupoids as continous groupoid functors

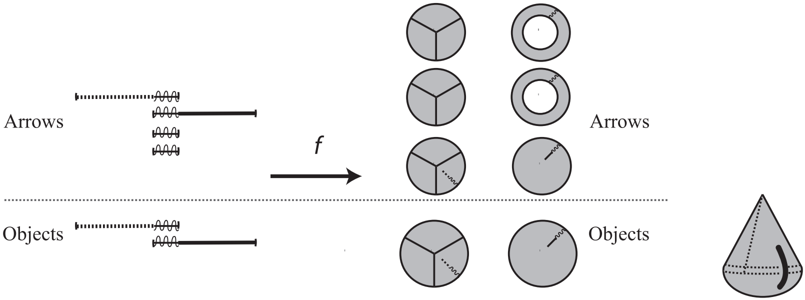
$$f_0 : \mathcal{G}_0 \to \mathcal{H}_0$$

and

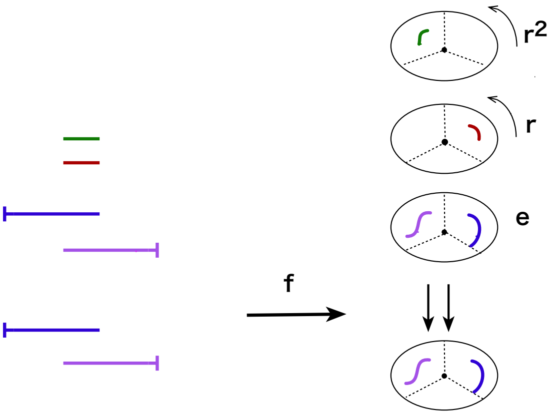
$$f_1 : \mathcal{G}_1 \to \mathcal{H}_1$$

such that if $g : x \to y$ then $f_1(g) : f_0(x) \to f_0(y)$

Example



Example



2-Cells

A **2-cell**

$$\alpha\colon f \Rightarrow f'\colon \mathcal{G} \rightrightarrows \mathcal{H}$$

is a continuous natural transformation.
This means a continuous map

$$\alpha\colon \mathcal{G}_0 \to \mathcal{H}_1$$

such that

- $s \circ \alpha = f_0$ and $t \circ \alpha = f_0'$;
- (**naturality**) the squares commute

$$\begin{tikzcd}
f_0(sg) \arrow[r, "f_1(g)"] \arrow[d, "\alpha(sg)"'] & f_0(tg) \arrow[d, "\alpha(tg)"] \\
f_0'(sg) \arrow[r, "f_1'(tg)"'] & f_0'(tg)
\end{tikzcd}$$

2-Cells and Underlying Maps

- If $\alpha\colon \varphi \Rightarrow \psi\colon \mathcal{G} \to \mathcal{H}$ then φ and ψ induce the same maps between the underlying spaces.

- It is possible that $f, f'\colon \mathcal{G} \rightrightarrows \mathcal{H}$ induce the same map on the underlying spaces without being related by a 2-cell.

The Groupoid **GMap**$(\mathcal{G}, \mathcal{H})$

Let $\mathcal{G}$ and $\mathcal{H}$ be topological groupoids.
We can form a mapping groupoid **GMap**$(\mathcal{G}, \mathcal{H})$ of maps $\mathcal{G}$ to $\mathcal{H}$ where

- $M_0 =$ continuous functors
- $M_1 =$ continuous natural transformations

Topology on $\mathbf{GMap}(\mathcal{G}, \mathcal{H})$

- Space of functors

$$\mathbf{GMap}(\mathcal{G}, \mathcal{H})_0 \subset \mathbf{Map}(\mathcal{G}_0, \mathcal{H}_0) \times \mathbf{Map}(\mathcal{G}_1, \mathcal{H}_1)$$

with elements

$$(f_0, f_1).$$

- Space of natural transformations

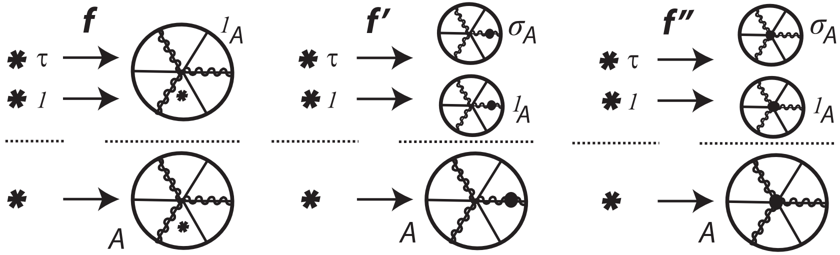
$$\mathbf{GMap}(\mathcal{G}, \mathcal{H})_1 \subset \mathbf{GMap}(\mathcal{G}, \mathcal{H})_0 \times \mathbf{Map}(\mathcal{G}_0, \mathcal{H}_1) \times \mathbf{GMap}(\mathcal{G}, \mathcal{H})_0,$$

with elements

$$(f_0, f_1, \alpha, g_0, g_1)$$

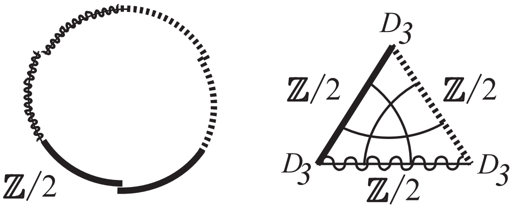
Example: **GMap**$(*_{\mathbb{Z}/2}, \mathbb{T})$

Maps from $*_{\mathbb{Z}/2}$ to the triangular billiard:



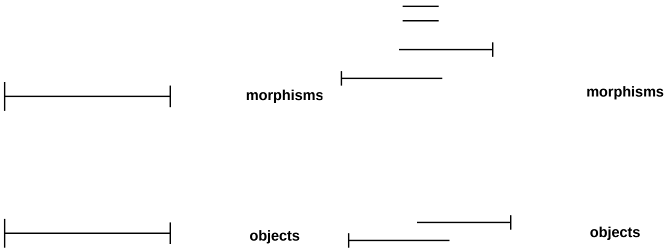
Example: **GMap**$(*_{\mathbb{Z}/2}, \mathbb{T})$

- We obtain a copy of the original orbigroupoid $\mathbb{T}$ together with a copy of the (trivial) $\mathbb{Z}/2$-circle, $S^1_{\mathbb{Z}/2}$,



From Orbigroupoids to Orbispaces

- The following two groupoids both represent the unit interval as orbispace



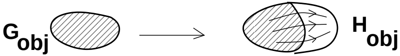
- They are not isomorphic in the category of orbigroupoids and groupoid homomorphisms.
- However, the groupoid homomorphism from the second to the first is an essential equivalence.

Essential Equivalences

- A morphism $f\colon \mathcal{G} \to \mathcal{H}$ is an **essential equivalence** when it is essentially surjective and fully faithful.
- It is **essentially surjective** when $\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \longrightarrow \mathcal{H}_0$ in

$$\begin{tikzcd}
\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \arrow[r] \arrow[d] & \mathcal{H}_1 \arrow[r, "t"] \arrow[d, "s"] & \mathcal{H}_0 \\
\mathcal{G}_0 \arrow[r, "f_0"'] & \mathcal{H}_0 &
\end{tikzcd}$$

is an **open surjection**.



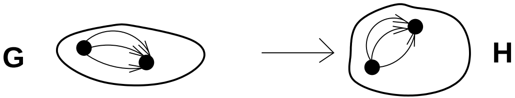
f may not be onto the objects of $\mathcal{H}$, but every object in $\mathcal{H}_0$ is isomorphic to an object in the image of $\mathcal{G}_0$.

Essential Equivalences

The morphism $f\colon \mathcal{G} \to \mathcal{H}$ is **fully faithful** when

$$\begin{tikzcd} G_1 \arrow[r, "\phi"] \arrow[d, "{(s,t)}"'] & H_1 \arrow[d, "{(s,t)}"] \\ G_0 \times G_0 \arrow[r, "\phi\times\phi"] & H_0 \times H_0 \end{tikzcd}$$

is a **pullback**,



The local isotropy structure is preserved.

Morita Equivalence

- The equivalence relation generated by the essential equivalences is called **Morita Equivalence**
- Orbigroupoids represent the same orbispace iff they are Morita equivalent
- To define a category of orbispaces, we use a **bicategory of fractions** to invert the essential equivalences

Generalized Maps

- Maps are **generalized maps** defined by spans

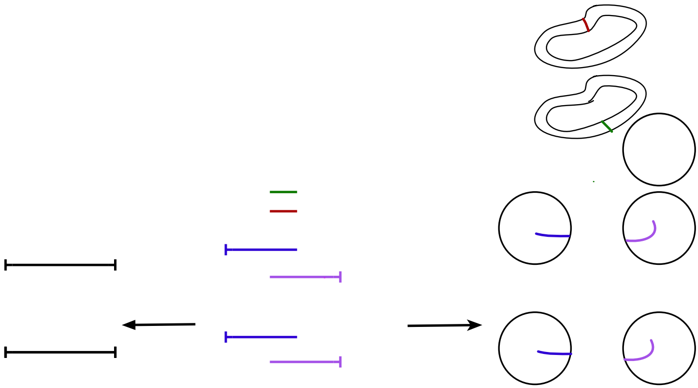
$$\mathcal{G} \xleftarrow{\upsilon} \mathcal{K} \xrightarrow{\varphi} \mathcal{H}$$

  where υ is an essential equivalence
- A **2-cell** between two generalized maps is an (equivalence class of) diagrams

$$\begin{tikzcd}
 & \mathcal{K} \arrow[dl, "\upsilon"'] \arrow[dr, "\varphi"] & \\
\mathcal{G} \arrow[r, phantom, "\alpha_1\Downarrow"] & \mathcal{L} \arrow[u, "\nu_1"'] \arrow[d, "\nu_2"] \arrow[r, phantom, "\alpha_2\Downarrow"] & \mathcal{H} \\
 & \mathcal{K}' \arrow[ul, "\upsilon'"] \arrow[ur, "\varphi'"'] &
\end{tikzcd}$$

  where $\upsilon\nu_1$ is an essential equivalence.

Example



Goals

1. Define a topology on the groupoid of **generalized** maps **OMap**$(\mathcal{G}, \mathcal{H})$
2. Show that composition induces a continuous map

   Show that **OMap** is Morita invariant (will follow from 2)
3. Show that **OMap**$(\mathcal{G}, \mathcal{H})$ is an orbigroupoid (will require compactness conditions)

Approaches

- We originally planned to define the topology on **OMap** as a pseudo-colimit
- We think this works (see Angel and Colman for path spaces)
- While trying to prove via pseudo-colimit, we looked closely at our bicategory of fractions
- We realized how nice this category actually was
- We switched to using its properties for a more direct approach

Goal 1: Defining a topology

- Problem: too many essential equivalences, we need a set
- Solution: Category Theory! [Bicategories of Fractions Edition]

Bicategories of Fractions

- We introduced weakened conditions for the existence of bicategories of fractions [omitted here]
- We defined a **cover** of a class of arrows $\mathfrak{W}$
- We use weakened conditions to prove that when $\mathfrak{V}$ covers $\mathfrak{W}$ then
$$\mathcal{B}(\mathfrak{W}^{-1}) \simeq \mathcal{B}(\mathfrak{V}^{-1}).$$
- For any given codomain, we can produce a **set** of arrows that cover the essential equivalences

A Cover of $\mathfrak{W}$

Definition

Let $\mathfrak{V} \subseteq \mathfrak{W}$ be two classes of arrows in a bicategory $\mathscr{B}$. The class $\mathfrak{V}$ is said to *cover* $\mathfrak{W}$ if for each arrow $w \in \mathfrak{W}$, there is an arrow v such that $wv \in \mathfrak{V}$.

$$\begin{tikzcd} & {} \arrow[dr, "w \in \mathfrak{W}", "\forall"'] & \\ {} \arrow[ur, dashed, "v", "\exists"'] \arrow[rr, dotted, "wv \in \mathfrak{V}"'] & & {} \end{tikzcd}$$

Essential Coverings

- open subsets $\mathcal{U}$ of $\mathcal{G}_0$ form an **essential covering** if the map $(j_{\mathcal{U}})_0 \colon \coprod_{U \in \mathcal{U}} U \to \mathcal{G}_0$ is essentially surjective
- Note that an essential covering does not necessarily cover all of $\mathcal{G}_0$, but it meets every orbit.

Essential Covering Maps

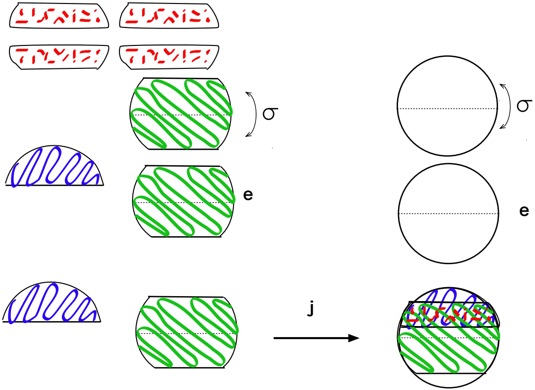
If $\mathcal{U}$ is an essential covering, we form a groupoid $\mathcal{G}^*(\mathcal{U})$ with a groupoid homomorphism $j_{\mathcal{U}}\colon \mathcal{G}^*(\mathcal{U}) \to \mathcal{G}$:

- $\mathcal{G}^*(\mathcal{U})_0 = \coprod_{U\in\mathcal{U}} U$;
- $(j_{\mathcal{U}})_0\colon \mathcal{G}^*(\mathcal{U})_0 \to \mathcal{G}_0$ is defined by inclusions on the connected components;
- $\mathcal{G}^*(\mathcal{U})_1$ is defined as the pullback,

$$\begin{tikzcd}
\mathcal{G}(\mathcal{U})_1 \arrow[r, "(j_{\mathcal{U}})_1"] \arrow[d, "(s{,}t)"'] & \mathcal{G}_1 \arrow[d, "(s{,}t)"] \\
\coprod_{U\in\mathcal{U}} U \times \coprod_{U\in\mathcal{U}} U \arrow[r, "(j_{\mathcal{U}})_0\times(j_{\mathcal{U}})_0"'] & \mathcal{G}_0 \times \mathcal{G}_0.
\end{tikzcd}$$

- This makes the map $j_{\mathcal{U}}\colon \mathcal{G}^*(\mathcal{U}) \to \mathcal{G}$ an essential equivalence.

Example



Theorem: Properties of Essential Covering Maps

- For any orbigroupoid $\mathcal{G}$, there is a **set** of (non-repeating) essential covering maps with codomain $\mathcal{G}$.
- The essential covering maps **cover** the essential equivalences between orbigroupoids.
- If $\mathfrak{W} =$ essential equivalences and $\mathfrak{C} =$ essential covering maps. Then,

$$\textbf{OrbiGrpds}(\mathfrak{W}^{-1}) \simeq \textbf{OrbiGrpds}(\mathfrak{C}^{-1})$$

Corollary 1

- Given two orbigroupoids $\mathcal{G}$ and $\mathcal{H}$, each orbimap

$$\mathcal{G} \xleftarrow[w]{} \mathcal{K} \xrightarrow[\varphi]{} \mathcal{H}$$

is isomorphic to one of the form,

$$\mathcal{G} \xleftarrow[j_{\mathcal{U}}]{} \mathcal{G}^*(\mathcal{U}) \xrightarrow[\psi]{} \mathcal{H}$$

where $\mathcal{U}$ is an essential covering

Corollary 2

- Any 2-cell from

$$\mathcal{G} \xleftarrow[j_{\mathcal{U}}]{} \mathcal{G}^*(\mathcal{U}) \xrightarrow[\varphi]{} \mathcal{H}$$

to

$$\mathcal{G} \xleftarrow[j_{\mathcal{V}}]{} \mathcal{G}^*(\mathcal{V}) \xrightarrow[\psi]{} \mathcal{H}$$

can be represented by a diagram of the form

$$\begin{tikzcd} & \mathcal{G}^*(\mathcal{U}) \arrow[dl, "j_{\mathcal{U}}"'] \arrow[dr, "\varphi"] & \\ \mathcal{G} \arrow[r, phantom, "\alpha"] & \mathcal{G}^*(\mathcal{W}) \arrow[u, "j^{\mathcal{W}}_{\mathcal{U}}"'] \arrow[d, "j^{\mathcal{W}}_{\mathcal{V}}"] \arrow[r, phantom, "\beta"] & \mathcal{H} \\ & \mathcal{G}^*(\mathcal{V}) \arrow[ul, "j_{\mathcal{V}}"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

The essential covering $\mathcal{W}$ can be viewed as an essential refinement of $\mathcal{U}$ and $\mathcal{V}$.

Defining the Topology

Space of Objects:

- Let

$$\mathbf{CMap}(\mathcal{G}^*(\mathcal{U}), \mathcal{G}) \subseteq \mathbf{GMap}(\mathcal{G}^*(\mathcal{U}), \mathcal{G})$$

  the full subgroupoid on essential coverings, with the subspace topology

- Define

$$\mathbf{OMap}(\mathcal{G}, \mathcal{H})_0 = \coprod_{\mathcal{U}} \mathbf{CMap}(\mathcal{G}^*(\mathcal{U}), \mathcal{G})_0 \times \mathbf{GMap}(\mathcal{G}^*(\mathcal{U}), \mathcal{H})_0,$$

  where the coproduct is taken over all non-repeating essential covers of $\mathcal{G}_0$.

Defining the Topology

Space of Arrows: will be a **quotient space** of the space of diagrams

$$\begin{tikzcd}
 & \mathcal{G}^*(\mathcal{U}) \arrow[dl, "j_{\mathcal{U}}"'] \arrow[dr, "\varphi"] & \\
\mathcal{G} \arrow[r, phantom, "\alpha"] & \mathcal{G}^*(\mathcal{W}) \arrow[u, "j_{\mathcal{U}}^{\mathcal{W}}"'] \arrow[d, "j_{\mathcal{V}}^{\mathcal{W}}"] \arrow[r, phantom, "\beta"] & \mathcal{H} \\
 & \mathcal{G}^*(\mathcal{V}) \arrow[ul, "j_{\mathcal{V}}"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

since 2-cells are defined by equivalence classes of diagrams

Goal 2: Properties of Topology

We next want to show:

- Topology is Hausdorff
- Composition is continuous
- **OMap** is Morita invariant

The Equivalence Relation

Given any two generalized maps

$$(w, f) = \left(\mathcal{G} \xleftarrow{w} \mathcal{G}^*(\mathcal{U}) \xrightarrow{f} \mathcal{H} \right)$$

and

$$(w', f') = \left(\mathcal{G} \xleftarrow{w'} \mathcal{G}^*(\mathcal{U}') \xrightarrow{f'} \mathcal{H} \right),$$

and **ANY** common essential refinement,

$$\begin{tikzcd} \mathcal{G}^*(\mathcal{V}) \arrow[r, "s_{\mathcal{U},\mathcal{U}'}"] \arrow[d, "t_{\mathcal{U},\mathcal{U}'}"'] \arrow[dr, phantom, "\alpha"] & \mathcal{G}^*(\mathcal{U}) \arrow[d, "j_{\mathcal{U}}"] \\ \mathcal{G}^*(\mathcal{U}') \arrow[r, "j_{\mathcal{U}'}"'] & \mathcal{G} \end{tikzcd}$$

The Equivalence Relation

every 2-cell $(w, f) \Rightarrow (w', f')$ can be represented **uniquely** by a diagram of the form

$$\begin{tikzcd}
 & \mathcal{G}^*(\mathcal{U}) \arrow[dl, "w"'] \arrow[dr, "f"] & \\
\mathcal{G} \arrow[r, phantom, "\alpha_{w,w'}"] & \mathcal{G}^*(\mathcal{V}) \arrow[u, "s_{\mathcal{U},\mathcal{U}'}"'] \arrow[d, "t_{\mathcal{U},\mathcal{U}'}"] \arrow[r, phantom, "\beta"] & \mathcal{H} \\
 & \mathcal{G}^*(\mathcal{U}') \arrow[ul, "w'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

where $\alpha_{w,w'} = (\gamma'^{-1} j_{\mathcal{V}}^{\mathcal{U}'}) \cdot \alpha \cdot (\gamma j_{\mathcal{V}}^{\mathcal{U}})$ for **this particular chosen** common refinement

The Mapping Groupoid **OMap**$(\mathcal{G}, \mathcal{H})$

Let **OMap**$(\mathcal{G}, \mathcal{H})$ be the groupoid such that each object corresponds to a span,

$$\mathcal{G} \xleftarrow{w} \mathcal{G}^*(\mathcal{U}) \xrightarrow{f} \mathcal{H}$$

and each arrow corresponds to an equivalence class of diagrams,

$$\begin{tikzcd}
 & \mathcal{G}^*(\mathcal{U}) \arrow[dl, "j_{\mathcal{U}}"'] \arrow[dr, "\varphi"] & \\
\mathcal{G} \arrow[r, phantom, "\alpha"] & \mathcal{G}^*(\mathcal{W}) \arrow[u, "j^{\mathcal{W}}_{\mathcal{U}}"'] \arrow[d, "j^{\mathcal{W}}_{\mathcal{V}}"] \arrow[r, phantom, "\beta"] & \mathcal{H} \\
 & \mathcal{G}^*(\mathcal{V}) \arrow[ul, "j_{\mathcal{V}}"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

The Space of Arrows Redux

- Choose an essential common refinement

$$\begin{tikzcd} \mathcal{G}^*(\mathcal{W}_{\mathcal{U},\mathcal{U}'}) \arrow[r, "s_{\mathcal{U},\mathcal{U}'}"] \arrow[d, "t_{\mathcal{U},\mathcal{U}'}"'] \arrow[dr, phantom, "\alpha"] & \mathcal{G}^*(\mathcal{U}) \arrow[d, "j_{\mathcal{U}}"] \\ \mathcal{G}^*(\mathcal{U}') \arrow[r, "j_{\mathcal{U}'}"'] & \mathcal{G} \end{tikzcd}$$

  for each pair $\mathcal{U}, \mathcal{U}'$ of essential coverings of $\mathcal{G}_0$;
- Choose corresponding induced 2-cells α for each pair of essential covering maps as above.

The Space of Arrows

Then each element of the space of arrows is represented by a **unique** diagram of the form

$$\begin{tikzcd} & \mathcal{G}^*(\mathcal{U}) \arrow[dl, "w"'] \arrow[dr, "f"] & \\ \mathcal{G} \arrow[r, phantom, "\alpha_{w,w'}"] & \mathcal{G}^*(\mathcal{W}_{\mathcal{U},\mathcal{U}'}) \arrow[u, "s_{\mathcal{U},\mathcal{U}'}"'] \arrow[d, "t_{\mathcal{U},\mathcal{U}'}"] \arrow[r, phantom, "\beta"] & \mathcal{H} \\ & \mathcal{G}^*(\mathcal{U}') \arrow[ul, "w'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

And the subspace of these diagrams is a retract, hence gives the quotient topology

The Space of Arrows

Write $P_{\mathcal{U},\mathcal{U}'}$ for the pseudo pullback of groupoids,

$$\begin{tikzcd}
P_{\mathcal{U},\mathcal{U}'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathbf{GMap}(\mathcal{G}^*(\mathcal{U}), \mathcal{H}) \arrow[d, "s^*_{\mathcal{U},\mathcal{V}}"] \\
\mathbf{GMap}(\mathcal{G}^*(\mathcal{U}'), \mathcal{H}) \arrow[r, "t^*_{\mathcal{U},\mathcal{U}'}"'] & \mathbf{GMap}(\mathcal{G}^*(\mathcal{W}_{\mathcal{U},\mathcal{U}'}), \mathcal{H}).
\end{tikzcd}$$

Then,

$$\mathbf{OMap}(\mathcal{G}, \mathcal{H})_1 \cong \coprod_{\mathcal{U},\mathcal{U}'} \mathbf{CMap}(\mathcal{G}^*(\mathcal{U}), \mathcal{G})_0 \times \mathbf{CMap}(\mathcal{G}^*(\mathcal{U}'), \mathcal{G})_0 \times (P_{\mathcal{U},\mathcal{U}'})_0.$$

In particular, this space is Hausdorff.

Results: Composition

Proposition

Composition by a generalized map
$(w, f) = \mathcal{G} \xleftarrow{w} \mathcal{G}^*(\mathcal{U}) \xrightarrow{f} \mathcal{H}$ *induces continuous groupoid maps between mapping groupoids,*

$$(w, f)_* \colon \mathbf{OMap}(\mathcal{K}, \mathcal{G}) \to \mathbf{OMap}(\mathcal{K}, \mathcal{H})$$

and

$$(w, f)^* \colon \mathbf{OMap}(\mathcal{H}, \mathcal{L}) \to \mathbf{OMap}(\mathcal{G}, \mathcal{L}).$$

Results: Morita Invariance

Theorem

Whenever $\mathcal{G}$ and $\mathcal{G}'$ are Morita equivalent and $\mathcal{H}$ and $\mathcal{H}'$ are Morita equivalent, the corresponding mapping groupoids

$$\mathbf{OMap}(\mathcal{G},\mathcal{H}) \textit{ and } \mathbf{OMap}(\mathcal{G}',\mathcal{H}')$$

are Morita equivalent.

Goal 3: **OMap** as an orbigroupoid

- Just as for manifolds, we need $\mathcal{G}$ to be compact in an appropriate sense.
- $\mathcal{G}$ is called **orbit-compact** when its underlying space $\mathcal{G}_0/\mathcal{G}_1$ is compact.

Results: Compact Case

- We can restrict to covers that are **essentially compact**: finitely many connected open subsets of $\mathcal{G}_0$ such that the closure of each of these open subsets in $\mathcal{G}_0$ is compact.
- When $\mathcal{G}$ is orbit-compact, we can replace the essential covering maps by essentially compact covering maps and define $\mathbf{OMap}_c(\mathcal{G}, \mathcal{H})$

Results: Compact Case

Theorem

If $\mathcal{G}$ is orbit-compact,

- $\mathbf{OMap}_c(\mathcal{G}, \mathcal{H})$ *is étale and proper;*
- *The inclusion* $\mathbf{OMap}_c(\mathcal{G}, \mathcal{H}) \hookrightarrow \mathbf{OMap}(\mathcal{G}, \mathcal{H})$ *is an essential equivalence;*
- $\mathbf{OMap}_c(\mathcal{G}, \mathcal{H})$ *is Morita invariant in both variables*
- $\mathbf{OMap}_c(\mathcal{G}, \mathcal{H})$ *has universal properties to be a mapping object*

Example: $\mathbf{OMap}(*_{\mathbb{Z}/2}, \mathbb{T})$

- Since we know we only need invert essential covers, we see that $\mathbf{OMap}(*_{\mathbb{Z}/2}, \mathbb{T}) = \mathbf{GMap}(*_{\mathbb{Z}/2}, \mathbb{T})$